HKDSE Mathematics
Indefinite Integration

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1. Introduction
Integration, either definite or indefinite, is a large topic in HKDSE Extended Module 2. With the introduction of integrations by substitution and by parts, which were not required in HKCEE Additional Mathematics, to the syllabus, a much wider variety of questions can be set compared with the old syllabus. Therefore, students should pay much more attention to this topic, and try to master this topic as well as possible for good results in M2 examination.
This note will demonstrate the techniques in solving problems involving indefinite integration as detailed as possible. It will start from the basic problems, and gradually to the hardest problems which involve advanced techniques in integration.

2. What is Indefinite Integration?
→ Indefinite integration can be considered the ‘reverse’ process of differentiation.
→ In differentiation, we find the derivative of a function.
→ In indefinite integration, we find the primitive function of a function.
→ In simpler words, if the derivative of \( F(x) \) is \( f(x) \), then \( F(x) \) is a primitive function of \( f(x) \),
i.e. \( \frac{d}{dx}F(x) = f(x) \).
→ The primitive function of a function is not unique in nature.
Note that \( \frac{d}{dx} F(x) = \frac{d}{dx} [F(x) + 1] = \frac{d}{dx} [F(x) + 2] = f(x) \).
Then \( F(x), \ F(x) + 1 \) and \( F(x) + 2 \) are all primitive functions of \( f(x) \).
When we replace 1 or 2 by any other real constants, say \( \pi, \ e, \sqrt{2} \) or \( -10 \), we still get the same result.
→ From the above results, we see that there are infinitely many primitive functions for any integrable function.
For convenience, if \( \frac{d}{dx} F(x) = f(x) \), then we write \( \int f(x)dx = F(x) + C \).
→ The constant \( C \) is called the constant of integration, and it is arbitrary in nature.
→ The sign \( \int \) is called the integral sign, and \( f(x) \) is called the integrand.

Quick Example:
As \( \frac{d}{dx} \left( \frac{\sqrt{x}}{2} \right) = \frac{1}{4\sqrt{x}} \), we have \( \int \frac{1}{4\sqrt{x}}dx = \frac{\sqrt{x}}{2} + C \).

Figure 1 A diagrammatic representation of differentiation and indefinite integration
3. **Methods of Indefinite Integration**

The methods of indefinite integration will be introduced below.

3.1. **Elementary Integration**

The following shows some fundamental indefinite integration results, which can be obtained directly from differentiation.

1. \[ \int k \, dx = kx + C, \text{ where } k \text{ is a constant} \]
2. \[ \int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \text{ where } n \neq -1 \]
3. \[ \int \frac{1}{x} \, dx = \ln|x| + C, \text{ where } x \neq 0 \]
4. \[ \int e^{ax} \, dx = \frac{e^{ax}}{a} + C, \text{ where } a \neq 0 \]

From (4), when \( a = 1 \), we have

5. \[ \int e^x \, dx = e^x + C. \]

The following gives the proof of (3). The remaining is left to readers as an exercise.

**Proof:**

When \( x > 0 \), \( \ln|x| = \ln x \). We have \[ \frac{d}{dx} (\ln x) = \frac{1}{x}. \]

When \( x < 0 \), \( \ln|x| = \ln(-x) \). We have \[ \frac{d}{dx} [\ln(-x)] = \frac{1}{-x} (-1) = \frac{1}{x}. \]

Combining the results, we have \[ \int \frac{1}{x} \, dx = \ln|x| + C. \]

Furthermore, we have the following properties. They can also be easily proved by employing differentiation.

6. \[ \int k f(x) \, dx = k \int f(x) \, dx, \text{ where } k \text{ is a constant} \]
7. \[ \int [f(x) \pm g(x)] \, dx = \int f(x) \, dx \pm \int g(x) \, dx \]

Using (7) repeatedly, we have

8. \[ \int [f_1(x) \pm f_2(x) \pm \ldots \pm f_n(x)] \, dx = \int f_1(x) \, dx \pm \int f_2(x) \, dx \pm \ldots \pm \int f_n(x) \, dx \]
Example 1

Find \( \int \left( x^2 - \frac{1}{x} \right)^4 \, dx \). (HKDSE Sample Paper)

Solution:

\[
\int \left( x^2 - \frac{1}{x} \right)^4 \, dx = \int \left( (x^2)^4 + C_1^4 (x^2)^3 \left( -\frac{1}{x} \right) + C_2^4 (x^2)^2 \left( -\frac{1}{x} \right)^2 + C_3^4 (x^2) \left( -\frac{1}{x} \right)^3 + \left( -\frac{1}{x} \right)^4 \right) \, dx
\]

\[
= \int \left( x^8 - 4x^5 + 6x^2 - \frac{4}{x} + \frac{1}{x^4} \right) \, dx
\]

\[
= \int (x^8 - 4x^5 + 6x^2 - \frac{4}{x} + x^{-4}) \, dx
\]

\[
= \frac{x^9}{9} - \frac{4x^6}{6} + 6 \cdot \frac{x^3}{3} - 4 \ln|x| + \frac{x^{-3}}{-3} + C
\]

\[
= \frac{x^9}{9} - \frac{2x^6}{3} + 2x^3 - 4 \ln|x| - \frac{1}{3x^3} + C
\]

Note: Always add the constant \( C \) after indefinite integration!

To make it simple, you should add it whenever the integral sign \( \int \) disappears.

Example 2

Find (a) \( \int (\sqrt{x} + 1)^2 \, dx \) (b) \( \int 10^x \, dx \).

Solution:

(a) \( \int (\sqrt{x} + 1)^2 \, dx \)

\[
= \int (x + 2\sqrt{x} + 1) \, dx
\]

\[
= \int (x + 2x^{\frac{1}{2}} + 1) \, dx
\]

\[
= \frac{x^2}{2} + 2 \cdot \frac{2}{3} x^{\frac{3}{2}} + x + C
\]

\[
= \frac{x^2}{2} + \frac{4}{3} x^{\frac{3}{2}} + x + C
\]

(b) We first let \( 10^x = e^{ax} \), where \( a \) is a constant.

Solving, we have \( a = \ln 10 \).

\[
\int 10^x \, dx
\]

\[
= \int e^{ax} \, dx
\]

\[
= \frac{e^{ax}}{a} + C
\]

\[
= \frac{10^x}{\ln 10} + C
\]

Binomial Theorem

\[
\ln 10^x = \ln e^{ax}
\]

\[
x \ln 10 = ax
\]

\[
a = \ln 10
\]
3.2. Integration by Substitution

The above formulae are very limited in usage and cannot deal with most integrals like $\int \frac{x+1}{x-1} \, dx$ or $\int -\frac{2x}{\sqrt{x^2+1}} \, dx$. We need a new method called ‘integration by substitution' to deal with these integrals.

Let $u = g(x)$. Then, we have $\int f(g(x))g'(x) \, dx = \int f(u) \, du$.

The proof is given in the appendix of this note on p.46.

This method can be regarded as the ‘reverse’ of the chain rule in differentiation. Refer to the following illustration to see how we can apply this method to find integrals:

Consider $\int \frac{2x}{\sqrt{x^2+1}} \, dx$. We take $u = g(x) = x^2 + 1$. Then, we have $g'(x) = 2x$.

\[
\int \frac{2x}{\sqrt{x^2+1}} \, dx = \int \frac{1}{\sqrt{g(x)}} \cdot 2x \, dx \\
= \int \frac{1}{\sqrt{g(x)}} g'(x) \, dx \\
= \int u^{-\frac{1}{2}} \, du \\
= 2u^{\frac{1}{2}} + C \\
= 2\sqrt{x^2+1} + C 
\]

As seen from the above example, we can see that we express the integral in the form $\int f(g(x))g'(x) \, dx$ and transform it into the form $\int f(u) \, du$ for simpler calculation.

However, we usually use the following way:

Let $u = x^2 + 1$, then $du = 2x \, dx$, i.e. $dx = \frac{1}{2x} \, du$.

\[
\int \frac{2x}{\sqrt{x^2+1}} \, dx = \int \frac{2x}{\sqrt{u}} \cdot \frac{1}{2x} \, du \\
= \int u^{-\frac{1}{2}} \, du \\
= 2u^{\frac{1}{2}} + C \\
= 2\sqrt{x^2+1} + C 
\]

We use a suitable substitution $u = g(x)$ at the beginning. We differentiate it with respect to $x$ (w.r.t. $x$, in short) to obtain $du = g'(x) \, dx$. By rearranging terms, we get $dx = \frac{1}{g'(x)} \, du$. We put it into the original integral and express the whole integral in terms of the new variable $u$ only. Finally, we find the integral, and express the result in terms of the original variable $x$. 

Example 3 (2012 DSE)

(a) Find $\int \frac{x+1}{x} \, dx$.

(b) Using the substitution $u = x^2 - 1$, find $\int \frac{x^3}{x^2 - 1} \, dx$.

Solution:

(a) $\int \frac{x+1}{x} \, dx = \int (1 + \frac{1}{x}) \, dx = x + \ln|x| + C$

(b) Let $u = x^2 - 1$, then $du = 2x \, dx$, i.e. $dx = \frac{1}{2x} \, du$.

$$\int \frac{x^3}{x^2 - 1} \, dx = \frac{1}{2} \int \frac{u+1}{u} \, du$$

(by (a))

$$= \frac{1}{2} (u + \frac{1}{2} \ln|u| + C)$$

$$= \frac{1}{2} (x^2 - 1 + \frac{1}{2} \ln|x^2 - 1| + C)$$

$$= \frac{1}{2} x^2 + \frac{1}{2} \ln|x^2 - 1| + C'$, where $C' = C - \frac{1}{2}$

*Think About*:

(1) Why did we not include $-\frac{1}{2}$ in final answer and use a new constant $C''$ instead?

(2) Why and how is $x^2$ changed into $u + 1$?

Example 4

Suppose $x > 0$.

Find (a) $\int e^{ax+b} \, dx$ (b) $\int (2ax + b)e^{ax^2+bx+c} \, dx$ (c) $\int \frac{1}{x^2} e^{\frac{x^2}{2}} \, dx$ (d) $\int \frac{\ln x}{x} \, dx$.

Solution:

(a) $\int e^{ax+b} \, dx = \frac{1}{a} \int e^{ax+b} \, d(ax + b) = \frac{e^{ax+b}}{a} + C$

(b) $\int (2ax + b)e^{ax^2+bx+c} \, dx = \int e^{ax^2+bx+c} \, d(ax^2 + bx + c) = e^{ax^2+bx+c} + C$

(c) $\int \frac{1}{x^2} e^{\frac{x^2}{2}} \, dx = -\frac{1}{2} \int e^{\frac{x^2}{2}} \, d\left(\frac{2}{x}\right) = -\frac{1}{2} e^{\frac{x^2}{2}} + C$

(d) $\int \frac{\ln x}{x} \, dx = \int \ln x \, d(\ln x) = \frac{(\ln x)^2}{2} + C$
Example 5

Find \(\int x^2 (x + 1)^{2013} \, dx\).

Solution:

Let \( u = x + 1 \), then \( du = dx \).

\[
\int x^2 (x + 1)^{2013} \, dx = \int (u - 1)^2 u^{2013} \, du
\]

\[
= \int (u^2 - 2u + 1)u^{2013} \, du
\]

\[
= \int (u^{2015} - 2u^{2014} + u^{2013}) \, du
\]

\[
= \frac{u^{2016}}{2016} - \frac{2u^{2015}}{2015} + \frac{u^{2014}}{2014} + C
\]

\[
= \frac{(x + 1)^{2016}}{2016} - \frac{2(x + 1)^{2015}}{2015} + \frac{(x + 1)^{2014}}{2014} + C
\]

Example 6

Find (a) \(\int \frac{x + 1}{x - 1} \, dx\) (b) \(\int \frac{x^2}{x - 1} \, dx\).

Solution:

(a) \(\int \frac{x + 1}{x - 1} \, dx = \int \frac{x - 1 + 2}{x - 1} \, dx\)

\[
= \int \left( \frac{x - 1}{x - 1} + \frac{2}{x - 1} \right) \, dx
\]

\[
= \int dx + 2\int \frac{1}{x - 1} \, d(x - 1)
\]

\[
= x + 2 \ln|x - 1| + C
\]

(b) Let \( u = x - 1 \), then \( du = dx \).

\[
\int \frac{x^2}{x - 1} \, dx
\]

\[
= \int \frac{(u + 1)^2}{u} \, du
\]

\[
= \frac{u^2}{u} + 2u + \frac{1}{u} \, du
\]

\[
= \int (u + 2 + \frac{1}{u}) \, du
\]

\[
= \frac{u^2}{2} + 2u + \ln|u| + C
\]

\[
= \frac{(x - 1)^2}{2} + 2(x - 1) + \ln|x - 1| + C
\]

\[
= \frac{(x - 1)^2}{2} + 2x + \ln|x - 1| + C'
\]
Example 7 (Long Division & Partial Fractions)

(a) Let \[
\frac{x^6 - x^5 - x^4 - x^3 - 6x^2 + 8x + 1}{x^2 - 3x + 2} = f(x) + \frac{g(x)}{x^2 - 3x + 2},
\] where \( f(x) \) and \( g(x) \) are two polynomials with \( \deg f(x) \leq 4 \) and \( \deg g(x) < 2 \). Find \( f(x) \) and \( g(x) \).

(b) Let \[
\frac{g(x)}{x^2 - 3x + 2} \equiv \frac{A}{x - 1} + \frac{B}{x - 2},
\] where \( A \) and \( B \) are constants. Find \( A \) and \( B \).

(c) Hence, find \[
\int \frac{x^6 - x^5 - x^4 - x^3 - 6x^2 + 8x + 1}{x^2 - 3x + 2} \, dx.
\]

Solution:

(a) Here, we perform long division.

\[
\begin{array}{c|rrrrrrrrrrr}
1 & 1 & 2 & 3 & 4 & 6 & 8 & 1 \\
-3 & 1 & -1 & -1 & -6 & 8 & +1 \\
\hline
& 1 & -3 & +2 & +2 & -6 & +4 & +3 & -9 & +6 & +4 & -12 & +8 & +1 & +1
\end{array}
\]

Then, we have \[
\frac{x^6 - x^5 - x^4 - x^3 - 6x^2 + 8x + 1}{x^2 - 3x + 2} = x^4 + 2x^3 + 3x^2 + 4x + \frac{1}{x^2 - 3x + 2}.
\]

Thus, \( f(x) \equiv x^4 + 2x^3 + 3x^2 + 4x \) and \( g(x) \equiv 1 \).

(b) From (a), \( g(x) \equiv 1 \).

\[
\frac{1}{x^2 - 3x + 2} \equiv \frac{A}{x - 1} + \frac{B}{x - 2}
\]

\[
1 \equiv A(x - 2) + B(x - 1)
\]

Put \( x = 1 \), we have \( 1 = A(1 - 2) + B(1 - 1) \), i.e. \( A = -1 \).

Put \( x = 2 \), we have \( 1 = A(2 - 2) + B(2 - 1) \), i.e. \( B = 1 \).

(c) From above, we have \[
\int \frac{x^6 - x^5 - x^4 - x^3 - 6x^2 + 8x + 1}{x^2 - 3x + 2} \, dx = \int \left( x^4 + 2x^3 + 3x^2 + 4x + \frac{1}{x - 2} - \frac{1}{x - 1} \right) \, dx
\]

\[
= \frac{x^5}{5} + \frac{x^4}{2} + 2x^3 + 3x^2 + 4x + \ln|x - 2| - \ln|x - 1| + C
\]

Points to note:

1. The notation ‘deg’ is used to refer to the degree of a polynomial. For instance, let \( h(x) = x^5 - 2x^2 + 1 \), then \( \deg h(x) \) is equal to 5.

2. This type of long division without variables written is known as the method of detached coefficients.

3. The method employed in (b), i.e. to break an algebraic fraction into sum of fractions with denominators of smaller degrees, is called resolving an algebraic fraction into partial fractions.
Example 8 (Reduction Formula)
Let \( y = x(\ln x)^n \), where \( x > 0 \).

(a) Find \( \frac{dy}{dx} \).

(b) Let \( I_n = \int (\ln x)^n \, dx \), where \( n \geq 0 \).

Using the result in (a), or otherwise, show that \( I_n = x(\ln x)^n - nI_{n-1} \)

(c) Hence, find \( I_4 \).

Solution:

(a) \( \frac{dy}{dx} = x \cdot n(\ln x)^{n-1} \cdot \frac{1}{x} + (\ln x)^n \cdot (1) = n(\ln x)^{n-1} + (\ln x)^n \)

(b) Integrating both sides w.r.t. \( x \), we have
\[
y = \int n(\ln x)^{n-1} \, dx + \int (\ln x)^n \, dx
\]
\[
x(\ln x)^n = n\int (\ln x)^{n-1} \, dx + \int (\ln x)^n \, dx
\]
\[
\int (\ln x)^n \, dx = x(\ln x)^n - n\left(\int (\ln x)^{n-1} \, dx\right)
\]
\[
I_n = x(\ln x)^n - nI_{n-1}
\]

(c) Using the result in (b) repeatedly, we have
\[
I_4 = x(\ln x)^4 - 4I_3
\]
\[
= x(\ln x)^4 - 4\left[x(\ln x)^3 - 3I_2\right]
\]
\[
= x(\ln x)^4 - 4x(\ln x)^3 + 12\left[x(\ln x)^2 - 2I_1\right]
\]
\[
= x(\ln x)^4 - 4x(\ln x)^3 + 12x(\ln x)^2 - 24\left[x\ln x - I_0\right]
\]
\[
= x(\ln x)^4 - 4x(\ln x)^3 + 12x(\ln x)^2 - 24x\ln x + 24\int (\ln x)^0 \, dx
\]
\[
= x(\ln x)^4 - 4x(\ln x)^3 + 12x(\ln x)^2 - 24x\ln x + 24x + C
\]

Notes:

1. The method employed in this question is to establish a reduction formula, which reduces an integral from a higher to lower power, with the same form. A reduction formula can be used repeatedly to lower the power of the integral, until it can be integrated easily. In this way, the integrals in certain forms, no matter how high the power is, can be found.

2. We can find out a reduction formula by differentiation. However, we usually use a technique called integration by parts to find them out. This will be introduced in Section 3.5. (Refer to Example 34 for details)
3.3. Integration of Trigonometric Functions
Trigonometric functions can be differentiated. Similarly, they can also be integrated.

3.3.1. Basic Integration of Trigonometric Functions
From differentiation, we have the following basic results.

(1) \( \int \sin x \, dx = -\cos x + C \)
(2) \( \int \cos x \, dx = \sin x + C \)
(3) \( \int \sec^2 x \, dx = \tan x + C \)
(4) \( \int \csc^2 x \, dx = -\cot x + C \)
(5) \( \int \sec x \tan x \, dx = \sec x + C \)
(6) \( \int \csc x \cot x \, dx = -\csc x + C \)

For other integrals of trigonometric functions, different trigonometric identities are useful.

Example 9
Find (a) \( \int \frac{\sin^2 x}{2} \, dx \) (b) \( \int \tan^2 x \, dx \).

Solution:
(a) \( \int \frac{\sin^2 x}{2} \, dx = \int \frac{1 - \cos x}{2} \, dx = \frac{1}{2} x - \frac{1}{2} \int \cos x \, dx = \frac{1}{2} x - \frac{1}{2} \sin x + C \)
(b) \( \int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx = \tan x - x + C \)

(Can you find \( \int \frac{\cos^2 x}{2} \, dx \) and \( \int \cot^2 x \, dx \)? Give them a try!)

3.3.2. Integration of Trigonometric Functions with Method of Substitution
The method of substitution can also be used in finding integrals of trigonometric functions.

Example 10
Find \( \int \sin 3x \cos x \, dx \).

Solution:
\[
\int \sin 3x \cos x \, dx = \frac{1}{2} \int [\sin(3x + x) + \sin(3x - x)] \, dx
= \frac{1}{2} \int (\sin 4x + \sin 2x) \, dx
= \frac{1}{2} \cdot \frac{1}{4} \int \sin 4x \, dx + \frac{1}{2} \cdot \frac{1}{2} \int \sin 2x \, dx
= \frac{-1}{8} \cos 4x - \frac{1}{4} \cos 2x + C
\]

Point to note:
Generally, we have the following formulas for constants \( a \) and \( b \), with \( a \neq 0 \):
\[
\int \sin(ax + b) \, dx = -\frac{1}{a} \cos(ax + b) + C
\]
\[
\int \cos(ax + b) \, dx = \frac{1}{a} \sin(ax + b) + C
\]
They can be proven easily by differentiation or method of substitution, with substitution \( u = ax + b \). They are widely accepted and are not required to be proven again in your calculations.
Example 11

(a) Prove that \( \tan x = \frac{1 - \cos 2x}{\sin 2x} \).

(b) Hence, or otherwise, find \( \int \frac{1 - \cos 2x}{\sin 2x} \, dx \).

Solution:

(a) \( \text{RHS} = \frac{1 - \cos 2x}{\sin 2x} = \frac{2}{2 \sin x \cos x} \cdot \frac{1 - \cos 2x}{2} = \frac{\sin^2 x}{\sin x \cos x} = \frac{\sin x}{\cos x} = \tan x = \text{LHS} \)

Thus, we have \( \tan x = \frac{1 - \cos 2x}{\sin 2x} \).

(b) \( \int \frac{1 - \cos 2x}{\sin 2x} \, dx = \int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\int \frac{1}{\cos x} \, d(\cos x) = -\ln|\cos x| + C \) (or \( \ln|\sec x| + C \))

Example 12

Find \( \int \sec x \, dx \).

Solution:

\[ \int \sec x \, dx \]

\[ = \int \sec x (\sec x + \tan x) \, dx \]

\[ = \int \frac{d(\sec x + \tan x)}{\sec x + \tan x} \quad (\text{Here, } u = \sec x + \tan x \text{ and } du = \sec x (\sec x + \tan x) \, dx.) \]

\[ = \ln|\sec x + \tan x| + C \]

Notes:

(1) Students are strongly recommended to recite the substitution used in Example 12, as well as the way to find this integral.

(2) Alternatively, you can find the integral using partial fractions as follows:

\[ \int \sec x \, dx = \int \frac{1}{\cos x} \, dx = \int \frac{\cos x}{\cos^2 x} \, dx = \int \frac{1}{1 - \sin^2 x} \, d(\sin x) = \int \frac{1}{1 - u^2} \, du \quad (\text{for } u = \sin x) \]

We can let \( \frac{1}{1 - u^2} = \frac{A}{1 + u} + \frac{B}{1 - u} \). Solving the simultaneous equations set up by comparing like terms, we can get \( A = \frac{1}{2} \) and \( B = \frac{1}{2} \). (Refer to Example 7 for details)

Then, \( \int \sec x \, dx = \frac{1}{2} \int \frac{1}{1 + u} \, du + \frac{1}{2} \int \frac{1}{1 - u} \, du = \frac{1}{2} \ln|1 + u| - \frac{1}{2} \ln|1 - u| + C = \frac{1}{2} \ln\left|\frac{1 + \sin x}{1 - \sin x}\right| + C \).

(3) Can you find \( \int \csc x \, dx \)? Give it a try! (Hint: Let \( u = \csc x + \cot x \)
Example 13 (Subsidiary Angle)

(a) Let \(0 < \theta < \frac{\pi}{2}\). By differentiating \(\ln(\sec \theta + \tan \theta)\), show that \(\int \sec \theta d\theta = \ln(\sec \theta + \tan \theta) + C\), where \(C\) is a constant.

(b) Let \(\cos \theta - \sin \theta = r \cos(\theta + \alpha)\), where \(r\) and \(\alpha\) are constants with \(r > 0\) and \(-\frac{\pi}{2} < \alpha < \frac{\pi}{2}\).

Find \(r\) and \(\alpha\).

(c) Hence, find \(\int \frac{1}{\cos \theta - \sin \theta} d\theta\).

Solution:

(a) \(\frac{d}{d\theta} \ln(\sec \theta + \tan \theta) = \frac{1}{\sec \theta + \tan \theta} (\sec \theta \tan \theta + \sec^2 \theta) = \sec \theta\)

Integrating both sides w.r.t. \(\theta\), we have

\(\int \sec \theta d\theta = \ln(\sec \theta + \tan \theta) + C\)

(b) \(\cos \theta - \sin \theta = r \cos(\theta + \alpha) = r \cos \theta \sin \alpha - r \cos \alpha \sin \theta\)

Comparing like terms, we have

\[
\begin{align*}
\sin \alpha &= 1 & (1) \\
-\cos \alpha &= -1 & (2)
\end{align*}
\]

(1) : \(\tan \alpha = 1\)

(2) : \(\tan \alpha = 1\)

We get \(\alpha = \frac{\pi}{4}\).

\((1)^2 + (2)^2 : r^2 (\sin^2 \alpha + \cos^2 \alpha) = 1^2 + 1^2\)

We get \(r = \sqrt{2}\).

(c) From (b), we have \(\cos \theta - \sin \theta = \sqrt{2} \cos \left(\theta + \frac{\pi}{4}\right)\).

\[
\int \frac{1}{\cos \theta - \sin \theta} d\theta = \frac{1}{\sqrt{2}} \int \frac{1}{\cos \left(\theta + \frac{\pi}{4}\right)} d\theta
\]

\[
= \frac{1}{\sqrt{2}} \int \sec \left(\theta + \frac{\pi}{4}\right) d\left(\theta + \frac{\pi}{4}\right)
\]

\[
= \frac{1}{\sqrt{2}} \ln \left[ \sec \left(\theta + \frac{\pi}{4}\right) + \tan \left(\theta + \frac{\pi}{4}\right) \right] + C
\]

Notes:

(1) The form \(r \sin(x \pm \alpha)\) or \(r \cos(x \pm \alpha)\) is called the subsidiary angle form. It expresses the sum (or difference) of two sine (or cosine) functions (with any non-zero coefficients) as a product of a non-zero real number and a sine (or cosine) function.

(2) Subsidiary angle form is NOT required in HKDSE M2 Examination. However, question format as in Example 13 is possible to appear in the examination.
Example 14

Find (a) \( \int \frac{\sin(\ln x)}{x} \, dx \) (b) \( \int \sin x \sec^2 (\cos x) \, dx \) (c) \( x \csc(x^2 + 1) \cot(x^2 + 1) \, dx \).

Solution:

(a) \( \int \frac{\sin(\ln x)}{x} \, dx \)

\[
= \int \sin(\ln x) d(\ln x)
= -\cos(\ln x) + C
\]

(b) \( \int \sin x \sec^2 (\cos x) \, dx \)

\[
= -\int \sec^2 (\cos x) d(\cos x)
= -\tan(\cos x) + C
\]

(c) \( \int x \csc(x^2 + 1) \cot(x^2 + 1) \, dx \)

\[
= \frac{1}{2} \int \csc(x^2 + 1) \cot(x^2 + 1) d(x^2 + 1)
= -\frac{1}{2} \csc(x^2 + 1) + C
\]

3.3.3. Integration of Special Types of Trigonometric Functions

The integrals of some trigonometric functions can be found easily by certain methods. (Here, \( p \) and \( q \) are positive integers.)

Type A: \( \int \sin^p x \cos^q x \, dx \)

Example 15

Find (a) \( \int \sin^5 x \cos x \, dx \) (b) \( \int \sin^6 x \cos^3 x \, dx \) (c) \( \int \sin^6 x \cos^4 x \, dx \).

Solution:

(a) \( \int \sin^5 x \cos x \, dx = \int \sin^5 x d(\sin x) = \frac{\sin^6 x}{6} + C \)

(b) \( \int \sin^6 x \cos^3 x \, dx \)

\[
= \int \sin^6 x \cdot \cos^2 x \cdot \cos x \, dx
= \int \sin^6 x (1 - \sin^2 x) d(\sin x)
= \int \sin^6 x d(\sin x) - \int \sin^8 x d(\sin x)
= \frac{\sin^7 x}{7} - \frac{\sin^9 x}{9} + C
\]
(c) We first express \( \sin^6 x \cos^4 x \) as a sum of \( \cos nx \), where \( n \) is an integer.

\[
\sin^6 x \cos^4 x = (\sin x \cos x)^4 \sin^2 x = \left( \frac{1}{2} \sin 2x \right)^4 \left( \frac{1 - \cos 2x}{2} \right)
\]

\[
= \frac{1}{32} \sin^4 2x (1 - \cos 2x)
\]

\[
= \frac{1}{32} \left( \frac{1 - \cos 4x}{2} \right)^2 (1 - \cos 2x)
\]

\[
= \frac{1}{128} (1 - 2 \cos 4x + \cos^2 4x)(1 - \cos 2x)
\]

\[
= \frac{1}{128} \left( 1 - 2 \cos 4x + \frac{1 + \cos 8x}{2} \right)(1 - \cos 2x)
\]

\[
= \frac{1}{256} (3 - 4 \cos 4x + \cos 8x)(1 - \cos 2x)
\]

\[
= \frac{1}{256} (3 - 3 \cos 2x - 4 \cos 4x + 4 \cos 2x \cos 4x + \cos 8x - \cos 2x \cos 8x)
\]

\[
= \frac{1}{256} \left( 3 - 3 \cos 2x - 4 \cos 4x + 2 \cos 2x + 2 \cos 6x + \cos 8x - \frac{1}{2} \cos 10x - \frac{1}{2} \cos 6x \right)
\]

\[
= \frac{1}{512} (6 - 2 \cos 2x - 8 \cos 4x + 3 \cos 6x + 2 \cos 8x - \cos 10x)
\]

\[
\int \sin^6 x \cos^4 x \, dx = \frac{1}{512} \int (6 - 2 \cos 2x - 8 \cos 4x + 3 \cos 6x + 2 \cos 8x - \cos 10x) \, dx
\]

\[
= \frac{1}{512} \left( 6x - \sin 2x - 2 \sin 4x + \frac{1}{2} \sin 6x + \frac{1}{4} \sin 8x - \frac{1}{10} \sin 10x \right) + C
\]

\[
= \frac{3}{256} x - \frac{1}{512} \sin 2x - \frac{1}{256} \sin 4x + \frac{1}{1024} \sin 6x + \frac{1}{2048} \sin 8x - \frac{1}{5120} \sin 10x + C
\]

Notes:

(1) If at least one of \( p \) or \( q \) is odd, we use the substitution \( u = \sin x \) or \( u = \cos x \), in order to convert the integral into a form only with the substitution used. In Examples 13(a) and 13(b), we used \( u = \sin x \) as the substitution and the whole integral was expressed in terms of \( \sin x \).

(2) If both \( p \) and \( q \) are even, we use the identities \( \sin^2 x = \frac{1 - \cos 2x}{2} \) and \( \cos^2 x = \frac{1 + \cos 2x}{2} \) repeatedly to lower the power of the integral. The integral is eventually expressed as a sum of sine and cosine functions, and can then be integrated easily.
**Type B:** \[ \int \tan^p x \sec^q x \, dx \]

**Example 16**

Find (a) \[ \int \tan^5 x \sec^2 x \, dx \]  (b) \[ \int \tan^2 x \sec^4 x \, dx \]  (c) \[ \int \tan^3 x \sec^5 x \, dx \].

**Solution:**

(a) \[ \int \tan^5 x \sec^2 x \, dx \]

\[
= \int \tan^5 x \, d(\tan x) \\
= \frac{\tan^6 x}{6} + C
\]

(b) \[ \int \tan^2 x \sec^4 x \, dx \]

\[
= \int \tan^2 x \sec^2 x \cdot \sec^2 x \, dx \\
= \int \tan^2 x(\tan^2 x + 1) \, d(\tan x) \\
= \int \tan^4 x \, d(\tan x) + \int \tan^2 x \, d(\tan x) \\
= \frac{\tan^5 x}{5} + \frac{\tan^3 x}{3} + C
\]

(c) \[ \int \tan^3 x \sec^5 x \, dx \]

\[
= \int \tan^2 x \sec^4 x \cdot \sec x \tan x \, dx \\
= \int (\sec^2 x - 1) \sec^4 x \, d(\sec x) \\
= \int \sec^6 x \, d(\sec x) - \int \sec^4 x \, d(\sec x) \\
= \frac{\sec^7 x}{7} - \frac{\sec^5 x}{5} + C
\]

Notes:

(1) If \( q \) is even, we use the substitution \( u = \tan x \) and convert the whole integral in terms of \( \tan x \). Besides this, \( \sec^n x \) (where \( n \) is an even number) can be expressed in terms of \( \tan x \) using the relation \( \sec^2 x = 1 + \tan^2 x \).

(2) If both \( p \) and \( q \) are odd, we use the substitution \( u = \sec x \) and convert the whole integral in terms of \( \sec x \). Besides this, \( \tan^n x \) (where \( n \) is an even number) can be expressed in terms of \( \sec x \) using the relation \( \tan^2 x = \sec^2 x - 1 \).

(3) It is much more difficult to find the integral if \( p \) is even and \( q \) is odd. A new technique called ‘integration by parts’ is required, and it will be introduced in Section 3.5. (Refer to Example 33 on p. 33 for details.)

(4) Integrals in the form \[ \int \cot^p x \csc^q x \, dx \] are found in similar way.
**Type C:** \( \sin^n x \, dx, \cos^n x \, dx, \) etc.

The techniques employed are similar to those used in finding **Type A** and **Type B** integrals.

**Example 17**

(a) Find (i) \( \int \sin^4 x \, dx \)  (ii) \( \int \cos^5 x \, dx \)  (iii) \( \int \tan^4 x \, dx \)  (iv) \( \int \cot^5 x \, dx \).

(b) By differentiating \( \sin^{n-1} x \cos x \), show that \( \int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx \), where \( n \geq 2 \). Hence, find \( \int \sin^4 x \, dx \).

**Solution:**

(a) (i) \( \int \sin^4 x \, dx \)

\[
= \int \left(1 - \frac{\cos 2x}{2}\right)^2 \, dx
\]

\[
= \frac{1}{4} \int (1 - 2 \cos 2x + \cos^2 2x) \, dx
\]

\[
= \frac{1}{4} \int \left(1 - 2 \cos 2x + \frac{1 + \cos 4x}{2}\right) \, dx
\]

\[
= \frac{1}{8} \int (3 - 4 \cos 2x + \cos 4x) \, dx
\]

\[
= \frac{3}{8} x - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C
\]

(ii) \( \int \cos^5 x \, dx \)

\[
= \int (1 - \sin^2 x)^2 \cos x \, dx
\]

\[
= \int (1 - 2 \sin^2 x + \sin^4 x) \, d(sin x)
\]

\[
= \sin x - \frac{2}{3} \sin^3 x + \frac{1}{5} \sin^5 x + C
\]

(iii) \( \int \tan^4 x \, dx \)

\[
= \int (\sec^2 x - 1) \tan^2 x \, dx
\]

\[
= \int \sec^2 x \tan^2 x \, dx - \int \tan^2 x \, dx
\]

\[
= \int \tan^2 x \, d(\tan x) - \int (\sec^2 x - 1) \, dx
\]

\[
= \frac{\tan^3 x}{3} - \tan x + x + C
\]
(iv) \[ \int \cot^5 x \, dx \]

\[ = \int (\csc^2 x - 1)^2 \cdot \cot x \, dx \]
\[ = \int \csc^4 x \cot x \, dx - 2 \int \csc^2 x \cot x \, dx + \int \cot x \, dx \]
\[ = -\int \csc^3 x \, d(\csc x) + 2 \int \csc x \, d(\csc x) + \int \frac{\cos x}{\sin x} \, dx \]
\[ = -\csc^4 \frac{x}{4} + \csc^2 x + \frac{1}{\sin x} \, dx \]
\[ = -\csc^4 \frac{x}{4} + \csc^2 x + \ln|\sin x| + C \]

(b) Let \( y = \sin^{n-1} x \cos x \).
\[ \frac{dy}{dx} = \sin^{n-1} x (-\sin x) + (\cos x)(n-1)(\sin^{n-2} x)(\cos x) \]
\[ = -\sin^n x + (n-1)\sin^{n-2} x \cos^2 x \]
\[ = -\sin^n x + (n-1)\sin^{n-2} x (1 - \sin^2 x) \]
\[ = (n-1)\sin^{n-2} x - n\sin^n x \]

Integrating both sides w.r.t. \( x \), we have
\[ y = (n-1)\int \sin^{n-2} x \, dx - n\int \sin^n x \, dx \]
\[ n\int \sin^n x \, dx = (n-1)\int \sin^{n-2} x \, dx - \sin^n x \]
\[ \int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx \]
\[ \int \sin^4 x \, dx = -\frac{1}{4} \sin^3 x \cos x + \frac{3}{4} \int \sin^2 x \, dx \]
\[ = -\frac{1}{4} \sin^3 x \cos x + \frac{3}{4} \left[ -\frac{1}{2} \sin x \cos x + \frac{1}{2} \int (\sin x)^2 \, dx \right] \]
\[ = -\frac{1}{4} \sin^3 x \cos x - \frac{3}{8} \sin x \cos x + \frac{3}{8} x + C_2 \]

An integral may take up different forms. We obtain different results for the same integral \( \int \sin^4 x \, dx \).

If we subtract the result in (a)(i) from that in (b), we get:
\[ \left( -\frac{1}{4} \sin^3 x \cos x - \frac{3}{8} \sin x \cos x + \frac{3}{8} x \right) - \left( \frac{3}{8} x - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x \right) \]
\[ = -\frac{1}{4} \left( \frac{1}{2} \sin 2x \right) \left( \frac{1 - \cos 2x}{2} \right) - \frac{3}{16} \sin 2x + \frac{1}{4} \sin 2x - \frac{1}{16} \sin 2x \cos 2x \]
\[ = -\frac{1}{16} \sin 2x + \frac{1}{16} \sin 2x \cos 2x - \frac{3}{16} \sin 2x + \frac{1}{4} \sin 2x - \frac{1}{16} \sin 2x \cos 2x \]
\[ = 0 \]

It follows that \( C_1 = C_2 \). There is no difference between the two integrals, and they are solely equivalent.
3.4. Integration by Trigonometric Substitution

Integration of trigonometric functions can be helpful in finding certain indefinite integrals. Before learning how to do so, we have to learn about inverse trigonometric functions.

3.4.1. Inverse Trigonometric Functions

We have come across with a lot of trigonometric functions, say, \( y = \sin x \). To express \( x \) in terms of \( y \), we define \( x = \sin^{-1} y \) (or \( x = \arcsin y \)), where \( \sin^{-1} y \) is read as ‘arcsine \( y \).

(a) Principal Values

Consider \( y = \sin x \). For \( y = \frac{\sqrt{2}}{2} \), we may have \( x = \frac{\pi}{4} \), \( \frac{3\pi}{4} \), \( \frac{9\pi}{4} \),... (in fact, \( x = n\pi + (-1)^n \frac{\pi}{4} \), where \( n \) is an integer). A function should give one and only one output for every input (e.g. for \( x = \sin^{-1} y \), if \( x \) is a function of \( y \), then for any value of \( y \), there is one and only one corresponding value of \( x \). To make sure the relationship, we define an interval called ‘principal values’ to restrict the value of \( x \).

Refer to the following illustration:

Let \( y = \sin^{-1} x \). (Note that \( x \) and \( y \) is exchanged here, unlike the above example)

For arcsine functions, the principal values are \([-\frac{\pi}{2}, \frac{\pi}{2}]\) (i.e. \(-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\)).

When \( x = \frac{1}{2} \), we have \( y = \frac{\pi}{6} \), i.e. \( \sin^{-1} \frac{1}{2} = \frac{\pi}{6} \).

When \( x = -1 \), we have \( y = -\frac{\pi}{2} \), i.e. \( \sin^{-1}(-1) = -\frac{\pi}{2} \).

Note: Our calculators are user-friendly. When you enter inverse trigonometric functions, it will automatically show the principal value as the answer.

The principal values of different functions are shown in the table below:

<table>
<thead>
<tr>
<th>Function</th>
<th>Principal Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin^{-1} x )</td>
<td>([-\frac{\pi}{2}, \frac{\pi}{2}])</td>
</tr>
<tr>
<td>( \cos^{-1} x )</td>
<td>([0,\pi])</td>
</tr>
<tr>
<td>( \tan^{-1} x )</td>
<td>((-\frac{\pi}{2}, \frac{\pi}{2}))</td>
</tr>
<tr>
<td>( \csc^{-1} x )</td>
<td>([-\frac{\pi}{2}, -\frac{\pi}{2}) ) or ( (0, \frac{\pi}{2}) )</td>
</tr>
<tr>
<td>( \sec^{-1} x )</td>
<td>([-\frac{\pi}{2}, -\frac{\pi}{2}) ) or ( (0, \frac{\pi}{2}) )</td>
</tr>
<tr>
<td>( \cot^{-1} x )</td>
<td>((0,\pi))</td>
</tr>
</tbody>
</table>

Note: \( \cot \frac{\pi}{2} = 0 \).
Example 18

Find (a) \( \sin^{-1} 0.75 \) (b) \( \cos^{-1} \frac{\pi}{4} \) (c) \( \cot^{-1} 3.3 \), in \textbf{radian} measures, correct to 3 decimal places.

Solution:

(a) \( \sin^{-1} 0.75 \approx 0.848 \) 
   (i.e. \( \sin 0.848 \approx 0.75 \))

(b) \( \cos^{-1} \frac{\pi}{4} \approx 0.667 \) 
   (i.e. \( \cos 0.667 \approx \frac{\pi}{4} \))

(c) We cannot enter ‘\( \cot^{-1} \)’ directly into our calculators. Therefore, we use the following method.
   \[ \cot^{-1} 3.3 = \tan^{-1} \left( \frac{1}{3.3} \right) \]
   \( \approx 0.294 \)  
   (i.e. \( \cot 0.294 \approx 3.3 \))

(b) \textbf{Graphs of Inverse Trigonometric Functions}

The following shows the graphs of the six trigonometric functions and those of their inverse functions for comparison:
Figure 2 The graphs of the six trigonometric functions and their inverse functions
3.4.2. Basic Integration by Trigonometric Substitution

Trigonometric substitutions are useful in finding integrals involving $a^2 - x^2$, $a^2 + x^2$ and $x^2 - a^2$, where $a$ is a constant. The following examples will illustrate how we use the substitution to find these integrals.

**Type A:** Integrals involving $a^2 - x^2$

We use the substitution $x = a \sin \theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

**Example 19**

Find $\int \frac{1}{\sqrt{25-x^2}} \, dx$.

**Solution:**

Let $x = 5 \sin \theta$, then $dx = 5 \cos \theta \, d\theta$.

$$\int \frac{1}{\sqrt{25-x^2}} \, dx = \int \frac{1}{\sqrt{25-(5 \sin \theta)^2}} \cdot 5 \cos \theta \, d\theta$$

$$= \int \frac{5 \cos \theta}{5 \sqrt{\cos^2 \theta}} \, d\theta$$

$$= \int \frac{\cos \theta}{\cos \theta} \, d\theta$$

$$= \int d\theta$$

$$= \theta + C$$

$$= \sin^{-1} \frac{x}{5} + C$$

Notes:

(1) In the range $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $\cos \theta$ is positive. As a result, $\sqrt{\cos^2 \theta} = \cos \theta$.

(Normally, we do not have to specify the range of $\theta$ in our calculations.)

(2) We can also use the substitution $x = 5 \cos \theta$ as follows:

Let $x = 5 \cos \theta$, then $dx = -5 \sin \theta \, d\theta$.

$$\int \frac{1}{\sqrt{25-x^2}} \, dx = \int \frac{1}{\sqrt{25-(5 \cos \theta)^2}} \cdot (-5 \sin \theta) \, d\theta$$

$$= -\int \frac{5 \sin \theta}{5 \sqrt{\sin^2 \theta}} \, d\theta$$

$$= -\int \frac{\sin \theta}{\sin \theta} \, d\theta$$

$$= -\int d\theta$$

$$= -\theta + C$$

$$= -\cos^{-1} \frac{x}{5} + C$$
Example 20

Find $\int \sqrt{9-x^2} \, dx$.

Solution:
Let $x = 3 \sin \theta$, then $dx = 3 \cos \theta \, d\theta$.

\[
\int \sqrt{9-x^2} \, dx = \int \sqrt{9-(3 \sin \theta)^2} \cdot 3 \cos \theta \, d\theta
\]
\[
= 9 \int \sqrt{1-\sin^2 \theta} \cdot \cos \theta \, d\theta
\]
\[
= 9 \int \cos \theta \, d\theta
\]
\[
= \frac{9}{2} \int (1 + \cos 2\theta) \, d\theta
\]
\[
= \frac{9}{2} \theta + \frac{9}{4} \sin 2\theta + C
\]

Now, we need to express the result in terms of $x$.
\[
\theta = \sin^{-1} \frac{x}{3}
\]
\[
\sin 2\theta = 2 \sin \theta \cos \theta
\]

[We construct a right-angled triangle as shown on the right such that $\sin \theta = \frac{x}{3}$.] 

[Length of adjacent side $= \sqrt{3^2 - x^2} = \sqrt{9-x^2}$]

[Then, we have $\cos \theta = \frac{\sqrt{9-x^2}}{3}$.] 

\[
\sin 2\theta = 2 \left( \frac{x}{3} \right) \left( \frac{\sqrt{9-x^2}}{3} \right) = \frac{2x\sqrt{9-x^2}}{9}
\]

\[
\int \sqrt{9-x^2} \, dx = \frac{9}{2} \sin^{-1} \frac{x}{3} + \frac{9}{4} \left( \frac{2x\sqrt{9-x^2}}{9} \right) + C
\]
\[
= \frac{9}{2} \sin^{-1} \frac{x}{3} + \frac{1}{2} x\sqrt{9-x^2} + C
\]

Notes:
(1) Sometimes, we may need to further perform integration of trigonometric functions after using a trigonometric substitution.
(2) The steps of finding $\cos \theta$ (enclosed by $[ ]$) can be skipped, as long as the right-angled triangle (with the lengths of three sides clearly indicated) is clearly drawn.
**Type B: Integrals involving** $a^2 + x^2$

We use the substitution $x = a \tan \theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

**Example 21**

It is given that $\int \sec x \, dx = \ln |\sec x + \tan x| + C$. (Refer to Example 12 for details)

Find (a) $\int \frac{1}{x^2 + 3} \, dx$ (b) $\int \frac{1}{\sqrt{x^2 + 3}} \, dx$.

**Solution:**

(a) Let $x = \sqrt{3} \tan \theta$, then $dx = \sqrt{3} \sec^2 \theta \, d\theta$.

\[
\int \frac{1}{x^2 + 3} \, dx = \int \frac{\sqrt{3} \sec^2 \theta}{(\sqrt{3} \tan \theta)^2 + 3} \, d\theta
\]
\[
= \int \frac{\sqrt{3} \sec^2 \theta}{3(\tan^2 \theta + 1)} \, d\theta
\]
\[
= \frac{\sqrt{3}}{3} \int \sec^2 \theta \, d\theta
\]
\[
= \frac{\sqrt{3}}{3} \theta + C_1
\]
\[
= \frac{\sqrt{3}}{3} \tan^{-1} \frac{x}{\sqrt{3}} + C_1
\]

(b) Let $x = \sqrt{3} \tan \theta$, then $dx = \sqrt{3} \sec^2 \theta \, d\theta$.

\[
\int \frac{1}{\sqrt{x^2 + 3}} \, dx = \int \frac{\sqrt{3} \sec^2 \theta}{\sqrt{(\sqrt{3} \tan \theta)^2 + 3}} \, d\theta
\]
\[
= \int \frac{\sqrt{3} \sec^2 \theta}{\sqrt{3}(\tan^2 \theta + 1)} \, d\theta
\]
\[
= \int \frac{\sec^2 \theta}{\sec \theta} \, d\theta
\]
\[
= \int \sec \theta \, d\theta
\]
\[
= \ln |\sec \theta + \tan \theta| + C''
\]
\[
= \ln \left| \frac{x^2 + 3}{\sqrt{3}} + \frac{x}{\sqrt{3}} \right| + C''
\]
\[
= \ln \left| \sqrt{x^2 + 3} + x \right| - \ln \sqrt{3} + C''
\]
\[
= \ln \left| \sqrt{x^2 + 3} + x \right| + C''', \text{ where } C''' = C'' - \ln \sqrt{3}
\]

Note:

As $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $\sqrt{\sec^2 \theta} = \sec \theta$, but not $-\sec \theta$. 

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Example 22

Find \( \int \frac{x^3}{\sqrt{x^2 + 4}} \, dx \) by (a) a trigonometric substitution (b) an algebraic substitution.

Solution:

(a) Let \( x = 2 \tan \theta \), then \( dx = 2 \sec^2 \theta \, d\theta \).

\[
\int \frac{x^3}{\sqrt{x^2 + 4}} \, dx = \int \frac{(2 \tan \theta)^3}{\sqrt{(2 \tan \theta)^2 + 4}} \cdot 2 \sec^2 \theta \, d\theta
\]

\[
= \int \frac{8 \tan^3 \theta}{2\sqrt{\sec^2 \theta}} \cdot 2 \sec^2 \theta \, d\theta
\]

\[
= \int 8 \tan^3 \theta \sec \theta \, d\theta
\]

\[
= 8 \int (\sec^2 \theta - 1) d(\sec \theta)
\]

\[
= \frac{8}{3} \sec^3 \theta - 8 \sec \theta + C
\]

\[
= \frac{8}{3} \left( \frac{\sqrt{x^2 + 4}}{2} \right)^3 - 8 \left( \frac{\sqrt{x^2 + 4}}{2} \right) + C
\]

\[
= \frac{1}{3} (x^2 + 4)^{\frac{3}{2}} - 4 \sqrt{x^2 + 4} + C
\]

\[
= \frac{1}{3} \sqrt{x^2 + 4} (x^2 - 8) + C
\]

(b) Let \( u = x^2 + 4 \), then \( du = 2x \, dx \).

\[
\int \frac{x^3}{\sqrt{x^2 + 4}} \, dx = \int \frac{x^3}{\sqrt{u}} \cdot \frac{1}{2x} \, du
\]

\[
= \frac{1}{2} \int \frac{u - 4}{u^{\frac{1}{2}}} \, du
\]

\[
= \frac{1}{2} \int (u^{\frac{1}{2}} - 4u^{-\frac{1}{2}}) \, du
\]

\[
= \frac{1}{3} u^{\frac{3}{2}} - 4u^{\frac{1}{2}} + C
\]

\[
= \frac{1}{3} (x^2 + 4)^{\frac{3}{2}} - 4(x^2 + 4)^{\frac{1}{2}} + C
\]

\[
= \frac{1}{3} \sqrt{x^2 + 4} (x^2 - 8) + C
\]

Notes:

An integral can be found by different methods. In this example, we used trigonometric substitution and algebraic substitution, and we still arrived at the same result.
**Type C:** Integrals involving \( x^2 - a^2 \)

We use the substitution \( x = a \sec \theta \), where \( 0 < \theta < \frac{\pi}{2} \text{ or } -\pi < \theta < -\frac{\pi}{2} \).

**Example 23**

It is given that \( \int \sec x \, dx = \ln|\sec x + \tan x| + C \). (Refer to Example 12 for details)

Find (a) \( \int \frac{1}{x\sqrt{x^2 - 9}} \, dx \) (b) \( \int \frac{1}{\sqrt{x^2 - 9}} \, dx \).

**Solution:**

(a) Let \( x = 3 \sec \theta \), then \( dx = 3 \sec \theta \tan \theta \, d\theta \).

\[
\int \frac{1}{x\sqrt{x^2 - 9}} \, dx = \int \frac{3 \sec \theta \tan \theta}{3 \sec \theta \sqrt{(3 \sec \theta)^2 - 9}} \, d\theta \\
= \int \frac{\tan \theta}{3 \sqrt{\tan^2 \theta}} \, d\theta \\
= \frac{1}{3} \int \tan \theta \, d\theta \\
= -\frac{\tan \theta}{3} + C \\
= \frac{1}{3} \sec^{-1} x + C_1
\]

(b) Let \( x = 3 \sec \theta \), then \( dx = 3 \sec \theta \tan \theta \, d\theta \).

\[
\int \frac{1}{\sqrt{x^2 - 9}} \, dx = \int \frac{3 \sec \theta \tan \theta}{\sqrt{(3 \sec \theta)^2 - 9}} \, d\theta \\
= \int \frac{\sec \theta \tan \theta}{\sqrt{\tan^2 \theta}} \, d\theta \\
= \int \sec \theta \, d\theta \\
= \ln|\sec \theta + \tan \theta| + C' \\
= \ln \left( \frac{x}{3} + \frac{\sqrt{x^2 - 9}}{3} \right) + C' \\
= \ln|x + \sqrt{x^2 - 9}| + C'' \text{, where } C'' = C' - \ln 3
\]

**Notes:**

In the range \( 0 < \theta < \frac{\pi}{2} \text{ or } -\pi < \theta < -\frac{\pi}{2} \), \( \sqrt{\tan^2 \theta} = \tan \theta \), but not \( -\tan \theta \).
3.4.3. Variations of Integration by Trigonometric Substitution

**Example 24**

Find \( \int \frac{\cos x}{1 + \sin^2 x} \, dx \).

**Solution:**

Let \( \sin x = \tan \theta \), then \( \cos x \, dx = \sec^2 \theta \, d\theta \).

\[
\int \frac{\cos x}{1 + \sin^2 x} \, dx = \int \frac{\sec^2 \theta}{1 + \tan^2 \theta} \, d\theta = \int d\theta = \theta + C = \tan^{-1}(\sin x) + C
\]

**Example 25 (Method of Completing the Square)**

Find \( \int \frac{1}{\sqrt{-x^2 + 12x - 20}} \, dx \).

**Solution:**

We first express \( -x^2 + 12x - 20 \) in the form of \( a^2 - (x - b)^2 \), where \( a \) and \( b \) are constants.

\[
-x^2 + 12x - 20 = -(x^2 - 12x) - 20 = -\left[ x^2 - 12x + \left(\frac{12}{2}\right)^2 \right] - 20 + \left(\frac{12}{2}\right)^2 = 16 - (x - 6)^2 = 4^2 - (x - 6)^2
\]

Let \( x - 6 = 4 \sin \theta \), then \( dx = 4 \cos \theta d\theta \).

\[
\int \frac{1}{\sqrt{-x^2 + 12x - 20}} \, dx = \int \frac{1}{\sqrt{4^2 - (x - 6)^2}} \, dx = \int \frac{4 \cos \theta}{\sqrt{4^2 - (4 \sin \theta)^2}} \, d\theta = \int \frac{\cos \theta}{\cos \theta} \, d\theta = \theta + C = \sin^{-1} \frac{x - 6}{4} + C
\]

**Notes:**

The method of completing the square is used in converting quadratic expressions into the form required for using trigonometric substitutions. We usually use this method when we encounter a quadratic denominator, while the numerator is a constant.
3.5. Integration by Parts

3.5.1. Introduction

There are still some integrals that cannot be found using the above techniques, e.g. \( \int x \sin x \, dx \), \( \int x \ln x \, dx \) and \( \int e^x \cos x \, dx \), etc. which are mostly in the form of the products of two functions. Here, we need a new technique called integration by parts to deal with these integrals.

Let \( u = f(x) \) and \( v = g(x) \) be two differentiable functions. Then, we have \( \int u \, dv = uv - \int v \, du \).

The proof is amazingly simple with the help of the product rule of differentiation.

**Proof:**

\[
\frac{d}{dx}(uv) = uv' + vu'
\]

Integrating both sides w.r.t. \( x \), we have

\[
uv = \int uv' \, dx + \int vu' \, dx
\]

\[
= \int u \, dv + \int v \, du
\]

i.e. \( \int u \, dv = uv - \int v \, du \).

The following example illustrates how we can use integration by parts to find an integral.

**Example 26**

Find \( \int \ln x \, dx \).

**Solution:**

Consider \( \int \ln x \, dx \).

Take \( u = \ln x \) and \( v = x \).

We have \( du = \frac{1}{x} \, dx \) and \( dv = dx \).

\[
\int \ln x \, dx = \int u \, dv = uv - \int v \, du = x \ln x - \int x \cdot \frac{1}{x} \, dx = x \ln x - \int dx = x \ln x - x + C
\]

In fact, it can be written in a simpler way:

\[
\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} \, dx = x \ln x - \int dx = x \ln x - x + C
\]
3.5.2. **Common Integrals involving Integration by Parts**

There are many integrals that can be found using integration by parts. Some of them are listed here.

**Type A:** \( \int x^n \sin x \, dx \) or \( \int x^n \cos x \, dx \)

**Example 27**

Find (a) \( \int x \sin x \, dx \) (b) \( \int x \cos x \, dx \) (c) \( \int x^2 \sin^2 \frac{x}{2} \, dx \).

**Solution:**

(a) \( \int x \sin x \, dx \)

\[
\begin{align*}
&= -\int x \, d(\cos x) \\
&= -x \cos x + \int \cos x \, dx \\
&= -x \cos x + \sin x + C
\end{align*}
\]

Here, \( u = x \), \( v = \cos x \), \( du = dx \) and \( dv = -\sin x \, dx \).

(b) \( \int x \cos x \, dx \)

\[
\begin{align*}
&= \int x \, d(\sin x) \\
&= x \sin x - \int \sin x \, dx \\
&= x \sin x + \cos x + C
\end{align*}
\]

Here, \( u = x \), \( v = \sin x \), \( du = dx \) and \( dv = \cos x \, dx \).

(c) \( \int x^2 \sin^2 \frac{x}{2} \, dx \)

\[
\begin{align*}
&= \frac{1}{2} \int x^2 (1 - \cos x) \, dx \\
&= \frac{1}{2} \int x^2 \, dx - \frac{1}{2} \int x^2 \cos x \, dx \\
&= \frac{x^3}{6} - \frac{1}{2} \int x^2 d(\sin x) \\
&= \frac{x^3}{6} - \frac{1}{2} x^2 \sin x + \frac{1}{2} \sin x \, dx(x^2) \\
&= \frac{x^3}{6} - \frac{1}{2} x^3 \sin x + \int x \sin x \, dx \\
&= \frac{x^3}{6} - \frac{1}{2} x^3 \sin x - x \cos x + \sin x + C
\end{align*}
\]

Use integration by parts repeatedly here (as in part (a)) to find \( \int x \sin x \, dx \).

Notes:

Integration by parts can be used repeatedly to lower the power of some integrals. In DSE M2 examination, however, you are required to use this technique at most twice only to find an integral.
Type B: \[ \int x^n \ln x \, dx \]

Example 28

Find \( \int x^n \ln x \, dx \) for (a) \( n \neq -1 \) (b) \( n = -1 \).

Solution:

(a) \[ \int x^n \ln x \, dx \]

\[
= \frac{1}{n+1} \int \ln x d(x^{n+1})
\]
\[
= \frac{1}{n+1} x^{n+1} \ln x - \frac{1}{n+1} \int x^{n+1} d(\ln x)
\]
\[
= \frac{1}{n+1} x^{n+1} \ln x - \frac{1}{n+1} \int x^n \, dx
\]
\[
= \frac{1}{n+1} x^{n+1} \ln x - \frac{1}{(n+1)^2} x^{n+1} + C
\]

(b) \[ \int x^{-1} \ln x \, dx \]

\[
= \int \frac{\ln x}{x} \, dx
\]
\[
= \int \ln x d(\ln x)
\]
\[
= \frac{(\ln x)^2}{2} + C
\]

*Think About*:

Can we find the integral by expressing \( \int x^n \ln x \, dx \) as \( x^{n+1} \ln x - \int x d(x^n \ln x) \)?

Type C: \[ \int x^n e^x \, dx \]

Example 29

Find (a) \( \int xe^x \, dx \) (b) \( \int x^2 e^{2x} \, dx \).

Solution:

(a) \[ \int xe^x \, dx \]

\[
= \int x d(e^x)
\]
\[
= xe^x - \int e^x \, dx
\]
\[
= xe^x - e^x + C
\]
\( (b) \int x^2 e^{2x} \, dx \)
\[
= \frac{1}{2} \int x^2 \, d(e^{2x}) \\
= \frac{1}{2} x^2 e^{2x} - \frac{1}{2} \int e^{2x} \, d(x^2) \\
= \frac{1}{2} x^2 e^{2x} - \int x e^{2x} \, dx \\
= \frac{1}{2} x^2 e^{2x} - \frac{1}{2} \int x d(e^{2x}) \\
= \frac{1}{2} x^2 e^{2x} - \frac{1}{2} x e^{2x} + \frac{1}{2} \int e^{2x} \, dx \\
= \frac{1}{2} x^2 e^{2x} - \frac{1}{2} x e^{2x} + \frac{1}{4} e^{2x} + C
\]

**Type D:** \( \int e^{ax} \sin b x \, dx \) or \( \int e^{ax} \cos b x \, dx \)

**Example 30**

Find (a) \( \int e^x \sin x \, dx \) (b) \( \int e^x \cos b x \, dx \), where \( a, b \neq 0 \).

**Solution:**

(a) \( \int e^x \sin x \, dx \)
\[
= -\int e^x d(\cos x) \\
= -e^x \cos x + \int \cos x \, d(e^x) \\
= -e^x \cos x + \int e^x \cos x \, dx \\
= -e^x \cos x + \int e^x d(\sin x) \\
= -e^x \cos x + e^x \sin x - \int \sin x \, d(e^x) \\
= -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx
\]
Rearranging terms, we have
\[
2 \int e^x \sin x \, dx = -e^x \cos x + e^x \sin x + C
\]
\[
\int e^x \sin x \, dx = -\frac{1}{2} e^x \cos x + \frac{1}{2} e^x \sin x + C', \text{ where } C' = \frac{1}{2} C
\]

Note: You can try to find the integral by first expressing \( \int e^x \sin x \, dx \) as \( \sin x \, d(e^x) \).

You will get the same result at last!
(b) \[ \int e^{ax} \cos bx \, dx \]
\[ = \frac{1}{a} \int \cos bxe^{ax} \, dx \]
\[ = \frac{1}{a} e^{ax} \cos bx - \frac{1}{a} \int e^{ax} d(\cos bx) \]
\[ = \frac{1}{a} e^{ax} \cos bx + \frac{b}{a} \int e^{ax} \sin bx \, dx \]
\[ = \frac{1}{a} e^{ax} \cos bx + \frac{b}{a^2} \int \sin bx \, dx \]
\[ = \frac{1}{a} e^{ax} \cos bx + \frac{b}{a^2} e^{ax} \sin bx - \frac{b^2}{a^2} \int e^{ax} d(\sin bx) \]
\[ = \frac{1}{a} e^{ax} \cos bx + \frac{b}{a^2} e^{ax} \sin bx - \frac{b^3}{a^2} \int e^{ax} \cos bx \, dx \]

Rearranging terms, we have
\[ \frac{a^2 + b^2}{a^2} \int e^{ax} \cos bx \, dx = \frac{1}{a} e^{ax} \cos bx + \frac{b}{a^2} e^{ax} \sin bx + C \]
\[ \int e^{ax} \cos bx \, dx = \frac{a}{a^2 + b^2} e^{ax} \cos bx + \frac{b}{a^2 + b^2} e^{ax} \sin bx + C', \text{ where } C' = \frac{a^2}{a^2 + b^2} C \]

3.5.3. Other Examples Involving Integration by Parts

Example 31

(a) Let \( y = \sin^{-1} x \). By considering \( \frac{dy}{dx} \), or otherwise, show that \( \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}} \).

(b) Hence, or otherwise, find (i) \( \int \sin^{-1} x \, dx \) (ii) \( \int (\sin^{-1} x)^2 \, dx \).

Solution:

(a) From \( y = \sin^{-1} x \), we have \( x = \sin y \).

Then, we have \( \frac{dx}{dy} = \cos y \).

By inverse function rule of differentiation, we also have \( \frac{dy}{dx} = \left( \frac{dx}{dy} \right)^{-1} = \frac{1}{\cos y} \).

Also note that \( \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2} \).

Thus, we have \( \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}} \).
Notes:

(1) For \( y = \sin^{-1} x \), \(-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\). In this range, \( \cos y \) is positive, and hence \( \cos y = \sqrt{1 - \sin^2 y} \), but not \( -\sqrt{1 - \sin^2 y} \).

(2) The result can also be arrived at by finding \( \int \frac{1}{\sqrt{1 - x^2}} \, dx \) using the substitution \( x = \sin \theta \).

(Refer to Example 19 for details)

(b) (i) There are two methods for finding this integral.

Method 1:
\[
\int \sin^{-1} x \, dx = x \sin^{-1} x - \int x d(\sin^{-1} x)
\]
\[
= x \sin^{-1} x - \int \frac{x}{\sqrt{1 - x^2}} \, dx
\]
\[
= x \sin^{-1} x + \frac{1}{2} \int \frac{1}{\sqrt{1 - x^2}} \, d(1 - x^2)
\]
\[
= x \sin^{-1} x + \sqrt{1 - x^2} + C
\]

Method 2:
Let \( u = \sin^{-1} x \). Then, we have \( x = \sin u \), i.e. \( dx = \cos u \, du \).

\[
\int \sin^{-1} x \, dx = \int u \cos u \, du
\]
\[
= \int ud(sin u)
\]
\[
= u \sin u - \int \sin u \, du
\]
\[
= u \sin u + \cos u + C
\]
\[
= (\sin^{-1} x) \sin(\sin^{-1} x) + \cos(\sin^{-1} x) + C
\]
\[
= x \sin^{-1} x + \sqrt{1 - x^2} + C
\]

(ii) Again, there are two methods for this integral.

Method 1:
\[
\int (\sin^{-1} x)^2 \, dx = x(\sin^{-1} x)^2 - \int x d(\sin^{-1} x)^2
\]
\[
= x(\sin^{-1} x)^2 - 2 \int \frac{x \sin^{-1} x}{\sqrt{1 - x^2}} \, dx
\]
\[
= x(\sin^{-1} x)^2 + 2 \int \sin^{-1} x \, dx(\sqrt{1 - x^2})
\]
\[
= x(\sin^{-1} x)^2 + 2 \sin^{-1} x \sqrt{1 - x^2} - 2 \sqrt{1 - x^2} \int d(\sin^{-1} x)
\]
\[
= x(\sin^{-1} x)^2 + 2 \sin^{-1} x \sqrt{1 - x^2} - 2 \sqrt{1 - x^2} \, dx
\]
\[
= x(\sin^{-1} x)^2 + 2 \sin^{-1} x \sqrt{1 - x^2} - 2 x + C
\]
Method 2:  
We also let  \( u = \sin^{-1} x \). Then, we have  \( x = \sin u \), i.e.  \( dx = \cos u \, du \).

\[
\int (\sin^{-1} x)^2 \, dx = \int u^2 \cos u \, du \\
= \int u^2 \, d(\sin u) \\
= u^2 \sin u - \int \sin u \, d(u^2) \\
= u^2 \sin u - 2 \int u \sin u \, du \\
= u^2 \sin u + 2 \int u \, d(\cos u) \\
= u^2 \sin u + 2u \cos u - 2 \int \cos u \, du \\
= u^2 \sin u + 2u \cos u - 2\sin u + C \\
= (\sin^{-1} x)^2 \sin(\sin^{-1} x) + 2 \sin^{-1} x \cos(\sin^{-1} x) - 2 \sin(\sin^{-1} x) + C \\
= x(\sin^{-1} x)^2 + 2 \sin^{-1} x \sqrt{1-x^2} - 2x + C
\]

**Example 32**

Find (a)  \( \int e^{\sqrt{\sin x}} \cos x \, dx \) for  \( 0 < x < \pi \)  
(b)  \( \int \sin x \, dx \) for  \( x > 0 \).

**Solution:**

(a) Let  \( u = \sqrt{\sin x} \), then  \( du = \frac{1}{2\sqrt{\sin x}} \cos x \, dx \).

\[
\int e^{\sqrt{\sin x}} \cos x \, dx = \int e^u \cdot 2udu \\
= 2 \int ue^u \, du \\
= 2 \int ud(e^u) \\
= 2ue^u - 2 \int e^u \, du \\
= 2ue^u - 2e^u + C \\
= 2(\sqrt{\sin x} - 1)e^{\sqrt{\sin x}} + C
\]

(b) Let  \( u = \ln x \), then  \( du = \frac{1}{x} \, dx \).

\[
\int \sin x \, dx = \int x \sin u \, du \\
= \int e^u \sin u \, du
\]

Repeating the steps in Example 30, we have  \( \int e^u \sin u \, du = -\frac{1}{2}e^u \cos u + \frac{1}{2}e^u \sin u + C \).

\[
\int \sin x \, dx = -\frac{1}{2}e^{\ln x} \cos (\ln x) + \frac{1}{2}e^{\ln x} \sin (\ln x) + C \\
= \frac{1}{2}x \sin (\ln x) - \frac{1}{2}x \cos (\ln x) + C
\]
**Example 33** (Finding \( \int \tan^p x \sec^q x \, dx \) for even \( p \) and odd \( q \))

It is given that \( \int \sec x \, dx = \ln|\sec x + \tan x| + C \). (Refer to Example 12 for details)

Find (a) \( \int \sec^3 x \, dx \)  
(b) \( \int \tan^2 x \sec x \, dx \)  
(c) \( \int \tan^2 x \sec^3 x \, dx \)  
(d) \( \int x^2 \sqrt{x^2 - 1} \, dx \).

**Solution:**

(a) \( \int \sec^3 x \, dx \)

\[
= \int \sec x (\tan^2 x + 1) \, dx \\
= \int \sec x \tan^2 x \, dx + \int \sec x \, dx \\
= \tan x \sec x + \ln|\sec x + \tan x| - \tan x \sec x + \ln|\sec x + \tan x| - \int \sec^3 x \, dx
\]

Rearranging terms, we have

\[
2 \int \sec^3 x \, dx = \sec x \tan x + \ln|\sec x + \tan x| + C_1
\]

\[
\int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C_2
\]

(b) \( \int \tan^2 x \sec x \, dx \)

\[
= \int (\sec^2 x - 1) \sec x \, dx \\
= \int \sec^3 x \, dx - \int \sec x \, dx \\
= \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| - \ln|\sec x + \tan x| + C_3
\]

\[
= \frac{1}{2} \sec x \tan x - \frac{1}{2} \ln|\sec x + \tan x| + C_3
\]

Alternatively,

\[
\int \tan^2 x \sec x \, dx
\]

\[
= \int \tan x \sec x \, dx \\
= \sec x \tan x - \int \sec x \tan x \, dx \\
= \sec x \tan x - \int \sec^3 x \, dx \\
= \sec x \tan x - \sec x \tan^2 x \, dx - \int \sec x \, dx \\
= \sec x \tan x - \ln|\sec x + \tan x| - \int \sec x \tan^2 x \, dx
\]

33
Rearranging terms, we have

\[ 2 \int \tan^2 x \sec x \, dx = \sec x \tan x - \ln |\sec x + \tan x| + c \]

\[ \int \tan^2 x \sec x \, dx = \frac{1}{2} \sec x \tan x - \frac{1}{2} \ln |\sec x + \tan x| + C_3 \]

(c) \[ \int \tan^2 x \sec^3 x \, dx \]

\[ = \int (\sec^2 x - 1) \sec x \, d(\tan x) \]

\[ = \sec^3 x \tan x - \int \sec x \, dx \tan x + \int \sec x \, dx \]

\[ = \sec^3 x \tan x - \int \tan^2 x \sec^3 x \, dx - \sec x \tan x + \int \tan^2 x \, dx \sec x \]

\[ = \sec^3 x \tan x - 3 \int \tan^2 x \sec^3 x \, dx - \sec x \tan x + \int \tan^2 x \, dx \sec x \]

\[ = \sec^3 x \tan x - \sec x \tan x + \frac{1}{2} \sec x \tan x - \frac{1}{2} \ln |\sec x + \tan x| - 3 \int \tan^2 x \sec^3 x \, dx \quad \text{(by (b))} \]

Rearranging terms, we have

\[ 4 \int \tan^2 x \sec^3 x \, dx = \sec^3 x \tan x - \frac{1}{2} \sec x \tan x - \frac{1}{2} \ln |\sec x + \tan x| + C_4 \]

\[ \int \tan^2 x \sec^3 x \, dx = \frac{1}{4} \sec^3 x \tan x - \frac{1}{8} \sec x \tan x - \frac{1}{8} \ln |\sec x + \tan x| + C_5 \]

(d) Let \( x = \sec \theta \), then \( dx = \sec \theta \tan \theta \, d\theta \).

\[ \int x^2 \sqrt{x^2 - 1} \, dx = \int \sec^2 \theta \sqrt{\sec^2 \theta - 1} \cdot \sec \theta \tan \theta \, d\theta \]

\[ = \int \tan^2 \theta \sec^3 \theta \, d\theta \]

\[ = \frac{1}{4} \sec^3 \theta \tan \theta - \frac{1}{8} \sec \theta \tan \theta - \frac{1}{8} \ln |\sec \theta + \tan \theta| + C_5 \]

\[ = \frac{1}{4} x^3 \sqrt{x^2 - 1} - \frac{1}{8} x \sqrt{x^2 - 1} - \frac{1}{8} \ln |x + \sqrt{x^2 - 1}| + C_5 \]

Notes:

(1) The technique of integration by parts followed by rearranging terms is very useful in finding integrals in the form \( \int \tan^p x \sec^q x \, dx \), where \( p \) is even and \( q \) is odd.

(2) Here is a challenging problem: find \( \int \sqrt{x^2 - 4x + 20} \, dx \). Try it out!

[Hint: Make good use of part (a) of Example 33 by using a suitable substitution!]

(For answers, please read the appendix of this note on p.46.)
**Example 34** (Reduction Formula)

(a) Prove the reduction formulas in Examples 8 and 17, i.e.

(i) \[ \int (\ln x)^n \, dx = x(\ln x)^n - n \int (\ln x)^{n-1} \, dx \quad \text{for} \quad n \geq 0 ; \]

(ii) \[ \int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx \quad \text{for} \quad n \geq 2 . \]

(b) Let \( I_n = \int \frac{dx}{(x^n + 1)^n} \), where \( m \) and \( n \) are positive integers.

Show that \( I_{n+1} = \frac{x}{mn(x^m + 1)^n} + \frac{mn-1}{mn} I_n \). Hence, or otherwise, find \( \int \frac{dx}{(x^2 + 1)^3} \).

**Solution:**

(a) (i) \[ \int (\ln x)^n \, dx \]

\[ = x(\ln x)^n - \int x d(\ln x)^n \]

\[ = x(\ln x)^n - n \int x (\ln x)^{n-1} \cdot \frac{1}{x} \, dx \]

\[ = x(\ln x)^n - n \int (\ln x)^{n-1} \, dx \]

(ii) \[ \int \sin^n x \, dx \]

\[ = -\int \sin^{n-1} x d(\cos x) \]

\[ = -\sin^{n-1} x \cos x + \int \cos x d(\sin^{n-1} x) \]

\[ = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos^2 x \, dx \]

\[ = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x (1 - \sin^2 x) \, dx \]

\[ = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^n x \, dx \]

Rearranging terms, we have

\[ (n-1+1) \int \sin^n x \, dx = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx \]

\[ \int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx \]

Notes:

(1) Rearranging terms is a very commonly used technique used in proving reduction formula using integration by parts.

(2) In proving reduction formulas involving trigonometric functions, the trigonometric identities \( \sin^2 x + \cos^2 x = 1 \), \( \tan^2 x + 1 = \sec^2 x \) and \( \cot^2 x + 1 = \csc^2 x \) are helpful on most occasions.
(b) \[ I_n \]

\[
= \frac{x}{(x^m + 1)^n} - \int x d\left(\frac{1}{(x^m + 1)^n}\right) \\
= \frac{x}{(x^m + 1)^n} + n \int \frac{x(mx^{m-1})}{(x^m + 1)^{n+1}} dx \\
= \frac{x}{(x^m + 1)^n} + mn \int \frac{x^m}{(x^m + 1)^{n+1}} dx \\
= \frac{x}{(x^m + 1)^n} + mnI_n - mnI_{n+1}
\]

Rearranging terms, we have

\[
mnI_{n+1} = \frac{x}{(x^m + 1)^n} + (mn - 1)I_n \\
I_{n+1} = \frac{x}{mn(x^m + 1)^n} + \frac{mn - 1}{mn} I_n
\]

First, we take \( m = 2 \) and \( n = 2 \).

\[
\int \frac{dx}{(x^2 + 1)^3} = \frac{x}{(2)(2)(x^2 + 1)^2} + \frac{(2)(2) - 1}{(2)(2)} \int \frac{dx}{(x^2 + 1)^2}
\]

For \( \int \frac{dx}{(x^2 + 1)^2} \), take \( m = 2 \) and \( n = 1 \).

\[
\int \frac{dx}{(x^2 + 1)^3} = \frac{x}{4(x^2 + 1)^2} + \frac{3}{4} \left[ \frac{x}{(2)(1)(x^2 + 1)} + \frac{(2)(1) - 1}{(2)(1)} \int \frac{dx}{x^2 + 1} \right] \\
= \frac{x}{4(x^2 + 1)^2} + \frac{3x}{8(x^2 + 1)} + \frac{3}{8} \int \frac{dx}{x^2 + 1}
\]

For \( \int \frac{dx}{x^2 + 1} \), let \( x = \tan \theta \), then \( dx = \sec^2 \theta d\theta \).

\[
\int \frac{dx}{x^2 + 1} = \int \frac{\sec^2 \theta d\theta}{\tan^2 \theta + 1} \\
= \int d\theta \\
= \theta + C_1 \\
= \tan^{-1} x + C_1
\]

Thus, we have

\[
\int \frac{dx}{(x^2 + 1)^3} = \frac{x}{4(x^2 + 1)^2} + \frac{3x}{8(x^2 + 1)} + \frac{3}{8} \tan^{-1} x + C.
\]

Note: The integral \( \int \frac{dx}{(x^2 + 1)^3} \) can also be found by letting \( x = \tan \theta \) at the very beginning.

The reduction formula can even be found by differentiation. Try them out!
3.6 Integration of Miscellaneous Functions

Some functions can be integrated with a combination of the above techniques.

3.6.1. Finding Integrals in the Form \( \int \frac{Ax^2 + Bx + C}{Dx^2 + Ex + F} \, dx \)

We follow the following steps:

(In the following, all capital letters represent constants.)

1. Express \( \int \frac{Ax^2 + Bx + C}{Dx^2 + Ex + F} \, dx \) as \( \int \frac{Hx + I}{Dx^2 + Ex + F} \, dx \).

2. Express \( \int \frac{Hx + I}{Dx^2 + Ex + F} \, dx \) as \( \int \frac{J}{Dx^2 + Ex + F} \, d(Dx^2 + Ex + F) + \int \frac{K}{Dx^2 + Ex + F} \, dx \).

3. Find \( \int G \, dx \), \( \int \frac{J}{Dx^2 + Ex + F} \, d(Dx^2 + Ex + F) \) and \( \int \frac{K}{Dx^2 + Ex + F} \, dx \) one by one.

The integral is then found.

Example 35

Find \( \int \frac{3x^2 - 4x - 5}{x^2 + 2x + 5} \, dx \).

Solution:

\[
\int \frac{3x^2 - 4x - 5}{x^2 + 2x + 5} \, dx = \int \frac{3(x^2 + 2x + 5) - 10x + 20}{x^2 + 2x + 5} \, dx = \int 3 \, dx - \int \frac{10x + 20}{x^2 + 2x + 5} \, dx
\]

\[
= \int 3 \, dx - \int \frac{5(2x + 2)}{x^2 + 2x + 5} \, dx - 10 \int \frac{1}{x^2 + 2x + 5} \, dx
\]

\[
= 3x - 5 \int \frac{1}{x^2 + 2x + 5} \, dx - 10 \int \frac{1}{(x + 1)^2 + 4} \, dx
\]

For \( \int \frac{1}{(x + 1)^2 + 4} \, dx \), let \( x + 1 = 2 \tan \theta \), then \( dx = 2 \sec^2 \theta \, d\theta \).

\[
\int \frac{1}{(x + 1)^2 + 4} \, dx = \int \frac{2 \sec^2 \theta}{(2 \tan \theta)^2 + 4} \, d\theta = \frac{1}{2} \int d\theta = \frac{1}{2} \theta + C = \frac{1}{2} \tan^{-1} \frac{x + 1}{2} + C
\]

This step can be replaced by long division.

\[
\frac{x^2 + 2x + 5}{x^2 + 2x + 5} = (x + 1)^2 + 4 > 0 \text{ for all real number } x
\]

Hence, absolute sign is not required here for \( \ln(x^2 + 2x + 5) \).

Thus, we have

\[
\int \frac{3x^2 - 4x - 5}{x^2 + 2x + 5} \, dx = 3x - 5 \ln(x^2 + 2x + 5) - 5 \tan^{-1} \frac{x + 1}{2} + C.
\]
Example 36

(a) Let \( \frac{1}{x^3 + 1} \equiv \frac{A}{x+1} + \frac{Bx + C}{x^2 - x + 1} \), where \( A, B \) and \( C \) are constants. Find \( A, B \) and \( C \).

(b) Hence, find \( \int \frac{1}{x^3 + 1} \, dx \).

Solution:

(a) From the question, we have

\[
\frac{1}{x^3 + 1} \equiv \frac{A(x^2 - x + 1) + (Bx + C)(x + 1)}{(x+1)(x^2 - x + 1)} = \frac{(A + B)x^2 + (B + C - A)x + (A + C)}{x^3 + 1}
\]

Hence, we have \( 1 \equiv (A + B)x^2 + (B + C - A)x + (A + C) \).

Comparing the coefficients of like terms, we have

\[
\begin{align*}
A + B &= 0 \\
B + C - A &= 0 \\
A + C &= 1
\end{align*}
\]

Solving the above system of linear equations, we have \( A = \frac{1}{3}, B = -\frac{1}{3} \) and \( C = \frac{2}{3} \).

(b) \( \int \frac{1}{x^3 + 1} \, dx = \frac{1}{3} \int \frac{1}{x+1} \, dx - \frac{1}{3} \int \frac{x-2}{x^2 - x + 1} \, dx \)

\[
= \frac{1}{3} \ln|x+1| - \frac{1}{6} \int \frac{2x-1}{x^2 - x + 1} \, dx + \frac{1}{6} \int \frac{3}{x^2 - x + 1} \, dx
\]

\[
= \frac{1}{3} \ln|x+1| - \frac{1}{6} \int \frac{1}{x^2 - x + 1} \, d(x^2 - x + 1) + \frac{1}{2} \int \frac{1}{x - \frac{1}{2}} \, dx + \frac{3}{4}
\]

For \( \int \frac{1}{x - \frac{1}{2}} \, dx \), let \( x - \frac{1}{2} = \frac{\sqrt{3}}{2} \tan \theta \), then \( dx = \frac{\sqrt{3}}{2} \sec^2 \theta \, d\theta \).

\[
\int \frac{1}{x - \frac{1}{2}} \, dx = \frac{\sqrt{3}}{2} \int \frac{\sec^2 \theta}{\left( \frac{\sqrt{3}}{2} \tan \theta \right)^2} \, d\theta = \frac{2}{\sqrt{3}} \int \sec^2 \theta \, d\theta = \frac{2}{\sqrt{3}} \theta + C_1 = \frac{2}{\sqrt{3}} \tan^{-1}\left( \frac{2x-1}{\sqrt{3}} \right) + C_1
\]

Thus, we have \( \int \frac{1}{x^3 + 1} \, dx = \frac{1}{3} \ln|x+1| - \frac{1}{6} \ln(x^2 - x + 1) + \frac{1}{\sqrt{3}} \tan^{-1}\left( \frac{2x-1}{\sqrt{3}} \right) + C \).
3.6.2. General $t$-substitution for integrals involving trigonometric functions

There are some integrals that involve trigonometric functions in the denominator and/or numerator in the integrand, e.g. $\int \frac{1}{1+\sin x} \, dx$ and $\int \frac{1}{\sin x + \cos x + 1} \, dx$, etc. To find these integrals, we can use a special method which makes the whole integral be in terms of the variable $t = \tan \frac{x}{2}$.

Before talking about how we can use this method, we first see how we can express the three basic trigonometric functions, $\sin x$, $\cos x$ and $\tan x$, in terms of $\tan \frac{x}{2}$.

**Example 37**

Let $t = \tan \frac{x}{2}$. Express $\sin x$, $\cos x$ and $\tan x$ in terms of $t$.

**Solution:**

$$
\sin x = 2\sin \frac{x}{2} \cos \frac{x}{2} = 2 \frac{2 \tan \frac{x}{2}}{2} \cos \frac{x}{2} = \frac{2 \tan \frac{x}{2}}{1+t^2},
$$

$$
\cos x = 2\cos^2 \frac{x}{2} - 1 = \frac{2}{1-t^2} - 1 = \frac{2t}{1-t^2},
$$

$$
\tan x = \frac{2 \tan \frac{x}{2}}{1-\tan^2 \frac{x}{2}} = \frac{2t}{1-t^2}.
$$

**Notes:**

You may also consider constructing a right-angled triangle as shown after finding $\tan x$. In this case, $\sin x$ and $\cos x$ can also be found. However, this is NOT considered a good method, as you can only find $\sin x$ and $\cos x$ for $0 < x < \pi$ in this case. Using trigonometric identities would be a better method as the result is true for all real values of $x$.

There are also other methods to find out the way to express the three functions in terms of $t$. Interested readers can search for more information on the internet.
Example 38

Let \( I = \int \frac{1}{1 + \sin x} \, dx \). Using the substitution \( t = \tan \frac{x}{2} \), or otherwise, find \( I \).

Solution:

Let \( t = \tan \frac{x}{2} \).

\[
dt = \frac{1}{2} \sec^2 \frac{x}{2} \, dx
\]

\[
dx = \frac{2}{1 + \tan^2 \frac{x}{2}} \, dt = \frac{2}{1 + t^2} \, dt
\]

\[
I = \int \frac{1}{1 + \sin x} \, dx = \int \frac{1}{1 + \frac{2t}{1 + t^2}} \cdot \frac{2}{1 + t^2} \, dt
\]

(by Example 37)

\[
= \int \frac{2}{t^2 + 2t + 1} \, dt = 2 \int \frac{1}{(t + 1)^2} \, d(t + 1) = -\frac{2}{t + 1} + C = -\frac{2}{\tan \frac{x}{2} + 1} + C
\]

Notes:

(1) Readers should pay attention to the way we express \( dx \) in terms of \( t \).

(2) This integral can be especially found in an easier way:

\[
I = \int \frac{1}{\sin^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2} + \cos^2 \frac{x}{2}} \, dx = \int \frac{1}{(\sin \frac{x}{2} + \cos \frac{x}{2})^2} \, dx = \int \frac{\sec^2 \frac{x}{2}}{(\tan \frac{x}{2} + 1)^2} \, dx
\]

\[
= \int \frac{2}{(\tan \frac{x}{2} + 1)^2} \, d(\tan \frac{x}{2} + 1) = -\frac{2}{\tan \frac{x}{2} + 1} + C
\]

(3) You may also consider:

\[
\int \frac{1}{1 + \sin x} \, dx = \int \frac{1 - \sin x}{1 - \sin^2 x} \, dx = \int \frac{1 - \sin x}{\cos^2 x} \, dx = \int \sec^2 x \, dx + \int \frac{d(\cos x)}{\cos^2 x} = \tan x - \sec x + C
\]

*Think about*:

Can you show that the two answers obtained differ by a constant (1, in this case)?
Example 39

Find \( \int \frac{1}{2 \sin x - 3 \cos x + 5} \, dx \).

Solution:

Again, we let \( t = \tan \frac{x}{2} \), then \( dx = \frac{2}{1 + t^2} \, dt \). (Refer to Example 38)

\[
\int \frac{1}{2 \sin x - 3 \cos x + 5} \, dx = \int \frac{1}{2 \left( \frac{2t}{1 + t^2} \right) - 3 \left( \frac{1 - t^2}{1 + t^2} \right) + 5} \cdot \frac{2}{1 + t^2} \, dt
\]

\[
= \int \frac{2}{4t - 3 + 3t^2 + 5 + 5t^2} \, dt
\]

\[
= \int \frac{1}{4t^2 + 2t + 1} \, dt
\]

\[
= \int \frac{1}{4 \left( t^2 + \frac{1}{2} t + \frac{1}{16} \right)} + \frac{1}{4} \, dt
\]

\[
= \int \frac{1}{4 \left( t + \frac{1}{4} \right)^2 + \frac{3}{4}} \, dt
\]

\[
= \frac{1}{4} \int \frac{1}{\left( t + \frac{1}{4} \right)^2 + \frac{3}{16}} \, dt
\]

Let \( t + \frac{1}{4} = \frac{\sqrt{3}}{4} \tan \theta \), then \( dt = \frac{\sqrt{3}}{4} \sec^2 \theta \, d\theta \).

\[
\int \frac{1}{2 \sin x - 3 \cos x + 5} \, dx = \frac{1}{4} \cdot \frac{\sqrt{3}}{4} \int \frac{\sec^2 \theta}{\left( \frac{\sqrt{3}}{4} \tan \theta \right)^2 + \frac{3}{16}} \, d\theta
\]

\[
= \frac{1}{\sqrt{3}} \int d\theta
\]

\[
= \frac{1}{\sqrt{3}} \theta + C
\]

\[
= \frac{1}{\sqrt{3}} \tan^{-1} \left( \frac{4t + 1}{\sqrt{3}} \right) + C
\]

\[
= \frac{1}{\sqrt{3}} \tan^{-1} \left( \frac{4 \tan \frac{x}{2} + 1}{\sqrt{3}} \right) + C
\]

Notes:

It is common that we have to use a further substitution to find the integral obtained by letting

\( t = \tan \frac{x}{2} \), and the techniques in Section 3.6.1 are useful in most situations.
4. **Applications of Indefinite Integration**

Indefinite integration is useful in both mathematics and daily life applications.

A. **Applications in Coordinate Geometry**

It is known that $f'(x)$ represents the slope of the tangent of a point $(x,y)$ to the curve $y = f(x)$. By integrating $f'(x)$ with respect to $x$, it is expected that we can obtain the original curve $y = f(x)$.

However, is it the case? No!

Integrating $f'(x)$ w.r.t. $x$, we have

$$\int f'(x)\,dx = y + C = f(x) + C$$

We observe that there is an extra constant $C$. In fact, what we have obtained is a family of curves, which can be obtained by translating the curve of $y = f(x)$ upwards or downwards.

We can further observe that all these curves have the same $f'(x)$, i.e. the slope of tangent to all curves at point $(x,y)$ is the same.

In order to find out the equation of a specific curve instead of a family of curves, we need the coordinates of a specific point to find out the constant $C$. This restricts the possibility of curves to only one curve, but not the family of curves as mentioned.

Similarly, suppose $f''(x)$ is given, we have

$$\int f''(x)\,dx = f'(x) + C_1$$

$$\int [f'(x) + C_1]\,dx = f(x) + C_1x + C_2$$

To find $f(x)$ from $f''(x)$, we need to integrate twice. We also need the coordinates of two points on the curve.

By substituting them into $y = f(x) + C_1x + C_2$, we can find out the constants $C_1$ and $C_2$.

Finally, we can find out the original function $f(x)$.

Refer to the examples on the next page.
Example 40

The slope of any point \((x, y)\) on a curve \(Q\) is \(\frac{x}{\sqrt{1 + x^2}}\). It is known that \(Q\) passes through the origin.

Find the equation of \(Q\).

Solution:

From the question, we have \(\frac{dy}{dx} = \frac{x}{\sqrt{1 + x^2}}\).

Integrating both sides w.r.t. \(x\), we have

\[
y = \int \frac{x}{\sqrt{1 + x^2}} \, dx
\]

\[
= \frac{1}{2} \int \frac{1}{\sqrt{1 + x^2}} \, d(1 + x^2)
\]

\[
= \sqrt{1 + x^2} + C
\]

Put \((0,0)\) into \(y = \sqrt{1 + x^2} + C\), we have

\[
0 = \sqrt{1 + 0^2} + C
\]

\[
C = -1
\]

Thus, the equation of \(Q\) is \(y = \sqrt{1 + x^2} - 1\).

Example 41

For curve \(Y\), it is given that \(\frac{d^2y}{dx^2} = e^{2x}\). If \(Y\) passes through points \(A(0,1)\) and \(B(1,1)\), find the equation of curve \(Y\).

Solution:

From \(\frac{d^2y}{dx^2} = e^{2x}\), by integration, we have \(\frac{dy}{dx} = \int e^{2x} \, dx = \frac{1}{2} e^{2x} + C_1\).

By further integration, we have \(y = \int \left(\frac{1}{2} e^{2x} + C_1\right) \, dx = \frac{1}{4} e^{2x} + C_1 x + C_2\)

Put \((0,1)\) and \((1,1)\) into \(y = \frac{1}{4} e^{2x} + C_1 x + C_2\), we get \(C_2 = \frac{3}{4}\) and \(C_1 + C_2 = 1 - \frac{1}{4} e^{2}\).

Solving the above system of linear equations, we get \(C_1 = \frac{1 - e^2}{4}\) and \(C_2 = \frac{3}{4}\).

Thus, the equation of curve \(Y\) is \(y = \frac{1}{4} e^{2x} + \frac{1 - e^2}{4} x + \frac{3}{4}\).
B. Applications in Physics

Consider an object travelling in a straight line:

We define some terms as follows:
Displacement \( s \) is the distance of an object from its original position, with the consideration of direction.
Velocity \( v \) is the derivative of displacement with respect to time \( t \).
Acceleration \( a \) is the derivative of velocity with respect to time.

If we take right hand side as positive direction, If \( PQ = 3 \) m, then the displacement of \( P \) is 3 m. On the contrary, if we take left hand side as positive direction, then the displacement of \( P \) is \(-3\) m.

Note that \( \frac{dv}{dt} = \frac{d^2s}{dt^2} \).

By integrating \( a \) or \( v \) with respect to \( t \), we can obtain \( v \) and \( s \) respectively.

Example 42

The velocity of a particle travelling along a straight line is given by \( v(t) = 5 - 2t \), where \( v(t) \) is the velocity (in m/s) and \( t \) is time (in second). It is known that the particle passes through its original position after 4 seconds. Take right hand side as positive direction.
(a) Find the displacement of the particle after \( t \) seconds.
(b) What is the displacement of the particle when it is momentarily at rest?

Solution:
(a) Let the displacement of the particle after \( t \) seconds be \( s(t) \) m.

Then \( s(t) = \int v(t) dt = \int (5 - 2t) dt = 5t - t^2 + C \).

When \( t = 4 \), \( s(t) = 0 \).

\( 5(4) - (4)^2 + C = 0 \)

\( C = -4 \)

Thus, the displacement of the particle is \((-t^2 + 5t - 4)\) m.

(b) The particle is momentarily at rest when \( v(t) = 0 \), i.e. \( t = \frac{5}{2} \).

The displacement of particle = \(- \left( \frac{5}{2} \right)^2 + 5 \left( \frac{5}{2} \right) - 4 = \frac{9}{4} \) m
C. Other Applications

Other than in physics, in many situations, if the rate of change is given, we can find out the original quantity by integrating the rate with respect to time.

Example 43

The figure on the right shows an inverted circular cone with base radius 30 cm and height 40 cm. At the beginning, the cone was full of water. A small hole was then drilled at the apex of the cone and water is flowing out of the cone at a rate of \((2t + 1)\text{ cm}^3/\text{s}\), where \(t\) is time in second.

(a) Express the volume of water remaining in the cone after \(t\) seconds.

(b) After how many seconds will the cone be empty?

Correct your answer to the nearest second.

Solution:

(a) Let the volume of water remaining in the cone be \(V\) cm\(^3\).

Then, we have

\[
\frac{dV}{dt} = -(2t + 1)
\]

Negative sign is added since the volume of water is decreasing.

\[
V = - \int (2t + 1)dt = -t^2 - t + C
\]

When \(t = 0\), we have \(V = \frac{1}{3} \pi (30)^2 (40) = 12000 \pi \text{ cm}^3\).

Thus, the volume of water remaining in the cone after \(t\) seconds is \((12000 \pi - t - t^2)\text{ cm}^3\).

(b) When the cone is empty, \(V = 0\).

\[
12000 \pi - t - t^2 = 0
\]

\[
t^2 + t - 12000 \pi = 0
\]

\[
t = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(-12000 \pi)}}{2(1)}
\]

\[
= \frac{-1 + \sqrt{48000 \pi + 1}}{2} \text{ or } \frac{-1 - \sqrt{48000 \pi + 1}}{2} \quad \text{(rejected, } t > 0)\]

\[
\approx 194
\]

Thus, the required time is 194 seconds.

More applications of integration will be found in the topic ‘Definite integrals’.

The writer would like to thank Mr. Yue Kwok Choy for his kind reading and checking of this article.
Appendix 1
Proof of Method of Substitution in Integration

To prove:

Let \( u = g(x) \). Then, we have \( \int f(g(x))g'(x)dx = \int f(u)du \).

Consider the function \( y = F(g(x)) \), where \( F(g(x)) \) is a primitive function of \( f(g(x)) \).

Then \( \frac{dy}{dx} = F'(g(x)) g'(x) = f(g(x))g'(x) \).

Integrating both sides w.r.t. \( x \), we have \( y = \int f(g(x)) g'(x)dx \).

But \( \frac{dy}{du} = F'(u) = f(u) \).

Integrating both sides w.r.t. \( u \), we have \( y = \int f(u)du \).

Then, we have \( \int f(g(x))g'(x)dx = \int f(u)du \).

This completes this proof.

Appendix 2
Finding \( \int \sqrt{x^2 - 4x + 20}dx \)

\[ \int \sqrt{x^2 - 4x + 20}dx = \int \sqrt{(x-2)^2 + 4^2}dx \]

Now let \( x - 2 = 4 \tan \theta \), then \( dx = 4 \sec^2 \theta d\theta \).

\[ \int \sqrt{x^2 - 4x + 20}dx = \int \sqrt{(x-2)^2 + 4^2}dx = \int \sqrt{(4 \tan \theta)^2 + 4^2} \cdot 4 \sec^2 \theta d\theta \]

\[ = 16 \int \sec^3 \theta d\theta \]

\[ = 8 \sec x \tan x + 8 \ln |\sec x + \tan x| + C \text{ (by Example 33(a))} \]

\[ = 8 \left( \frac{\sqrt{x^2 - 4x + 20}}{4} \right) \left( \frac{x-2}{4} \right) + 8 \ln \left| \frac{\sqrt{x^2 - 4x + 20} + x - 2}{4} \right| + C \]

\[ = \frac{1}{2} (x-2)\sqrt{x^2 - 4x + 20} + 8 \ln \left| \frac{\sqrt{x^2 - 4x + 20} + x - 2}{4} \right| + C' \]