Points of inflection

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1. Definition

A point of inflection (point of inflexion) \( (x_0, f(x_0)) \) on a curve is a continuous point at which the function \( f(x) \) changes from convex (concave upward) to concave (concave downward) or vice versa as \( x \) passes through \( x_0 \).

2. Continuity of the function

If \( (x_0, f(x_0)) \) is a point of inflection of the function \( y = f(x) \), then the function is also continuous at that point. In other words, if \( (x_0, y_0) \) is not continuous, it must not be a point of inflection.

Example 1

\( y = \tan x \) is concave upward in the interval \( \left( \pi/2n, \pi/2(n+1) \right) \) and concave downward in the interval \( \left( \pi/2n, \pi/2(n+1) \right) \), where \( n \in \mathbb{Z} \).

However, at \( x = \pi/2n \), \( y = \tan x \) is undefined and therefore these points are not points of inflection.

Readers may check that \( (n\pi, 0) \) are points of inflection.

3. First derivative

A point of inflexion of the curve \( y = f(x) \) must be continuous point but need not be differentiable there.

Example 2

\[ y = f(x) = x^{1/3} \]

\[ f'(x) = \frac{1}{3x^{2/3}}, \quad f''(x) = -\frac{2}{9x^{5/3}} \]

For \( x < 0 \), \( f''(x) > 0 \) \( \Rightarrow \) concave upward

For \( x > 0 \), \( f''(x) < 0 \) \( \Rightarrow \) concave downward

Although \( f'(0) \) and \( f''(0) \) are undefined, \( (0, 0) \) is still a point of inflection.

In fact, \( y = f(x) = x^{1/3} \) is the inverse function of \( y = x^3 \). The latter function obviously has also a point of inflection at \( (0, 0) \).
4. Second derivative

Even the first derivative exists in certain points of inflection, the second derivative may not exist at these points.

**Example 3**

\[ y = f(x) = x^{5/3} - x, \]

\[ f'(x) = \frac{5}{3}x^{2/3} - 1, \quad f''(x) = -\frac{10}{9x^{1/3}} \]

If \( x < 0, \) \( f''(x) < 0 \) ⇒ concave downward.
If \( x > 0, \) \( f''(x) > 0 \) ⇒ concave upward.
\( f'(0) = -1 \) exists but \( f''(0) \) does not exist.
However, \((0, 0)\) is a point of inflection.

Setting the second derivative of a function to zero sometimes **cannot** find out all points of inflection.

**Example 4**

\[ y = f(x) = x^{2/3}(x - 1)^{1/3} \]

\[ f'(x) = \frac{3x - 2}{3x^{1/3}(x - 1)^{2/3}}, \quad f''(x) = -\frac{2}{9x^{4/3}(x - 1)^{5/3}} \]

By setting \( f''(x) = 0 \), we get no solution for points of inflections.

By setting the denominator of \( f''(x) \) to 0, we get \( x = 0 \) or \( 1 \).
There is no sign change for \( f''(x) = 0 \) as \( x \) goes through \( x = 0 \).
\( \therefore (0, 0) \) is not a point of inflection.
There is a sign change for \( f''(x) = 0 \) as \( x \) goes through \( x = 1 \).
\( \therefore (1, 0) \) is not a point of inflection.

5. Relative extremum

Points of inflection can also be the extremum points (maximum / minimum) at the same time.

**Example 4**

\[ y = f(x) = \frac{|x|}{(x + 1)^2} \]
For \( x \neq 1 \),
\[
f(x) = \begin{cases} \frac{x}{(x+1)^2}, & \text{for } x > 0 \\ -\frac{x}{(x+1)^2}, & \text{for } x < 0 \end{cases}
\]
\[
f'(x) = \begin{cases} \frac{1-x}{(x+1)^2}, & \text{for } x > 0 \\ \frac{x-1}{(x+1)^2}, & \text{for } x < 0 \end{cases}
\]
\[
f''(x) = \begin{cases} \frac{2x-4}{(x+1)^2}, & \text{for } x > 0 \\ \frac{4-2x}{(x+1)^2}, & \text{for } x < 0 \end{cases}
\]

If \( x < 0 \), \( f''(x) > 0 \) \( \Rightarrow \) concave upward.
If \( x > 0 \), \( f''(x) < 0 \) \( \Rightarrow \) concave downward.
\( \therefore (0, 0) \) is a point of inflection.

Note that \((0, 0)\) is also a relative minimum point.

The reader may check that:
1. \( \left(\frac{2}{9}, \frac{2}{3}\right) \) is another inflection point.
2. \( \left(\frac{1}{4}, \frac{1}{2}\right) \) is a relative maximum point.
3. Horizontal asymptote: \( y = 0 \). Vertical asymptote: \( x = -1 \).

**Example 5**
\[ y = f(x) = \frac{x|x+1|}{x+2} \]

For \( x \neq -2 \),
\[
f(x) = \begin{cases} \frac{x(x+1)}{x+2}, & \text{for } x > -1 \\ -\frac{x(x+1)}{x+2}, & \text{for } x < -1 \end{cases}
\]
\[
f'(x) = \begin{cases} \frac{x^2 + 4x + 2}{x+2}, & \text{for } x > -1 \\ -\frac{x^2 + 4x + 2}{x+2}, & \text{for } x < -1 \end{cases}
\]
\[
f''(x) = \begin{cases} \frac{4}{(x+2)^3}, & \text{for } x > -1 \\ -\frac{4}{(x+2)^3}, & \text{for } x < -1 \end{cases}
\]

If \( x < -1 \), \( f''(x) > 0 \) \( \Rightarrow \) concave downward.
If \( x > -1 \), \( f''(x) < 0 \) \( \Rightarrow \) concave upward.
\( \therefore (-1, 0) \) is a point of inflection.

Note that \((-1, 0)\) is also a relative minimum point.

The reader may check that:
1. \( \left(-2 + \sqrt{2}, -3 + 2\sqrt{2}\right) \) is a relative minimum point.
2. \( \left(-2 - \sqrt{2}, -3 + 2\sqrt{2}\right) \) is a relative minimum point.
3. Vertical asymptote is \( x = -2 \).
4. Oblique asymptotes are \( y = x - 1 \) (for the positive side) and \( y = -x + 1 \) (for negative side).

Exercise

Find the point(s) of inflection of each of the following curves:

1. \( y = f(x) = \frac{1}{3} \sqrt[3]{x^2(x + 6)} \)
2. \( x^3 - y^3 = 8 \)
3. \( y = f(x) = \frac{1}{3} x^2 + \frac{1}{3} x - 1 \)

Derivative help:

1. \( y' = \frac{x + 4}{x^{1/3}(x + 6)^{2/3}}, \quad y'' = -\frac{8}{x^{4/3}(x + 6)^{5/3}} \)
2. \( y' = \frac{x^2}{(x^3 - 8)^{2/3}}, \quad y'' = -\frac{16x}{(x^3 - 8)^{5/3}} \)
3. \( y' = \frac{2[x^{4/3} + (x^2 - 1)^{2/3}]}{3x^{1/3}(x^2 - 1)^{2/3}}, \quad y'' = -\frac{2[3x^{4/3} + x^{10/3} - (x^2 - 1)^{2/3} + x^2(x^2 - 1)^{2/3}]}{9x^{4/3}(x^2 - 1)^{5/3}} \)

Graphical help:

1.
2.
3.