**Theory of equations**

1. **(a)** Let \( p(x) = x^4 + 4x^3 - 8x^2 - 1 \), then \( p'(x) = 4x^3 + 12x^2 - 16x = 4(x + 4)(x - 1) \)

   \[ x = -4, 0, 1 \] are roots of the equation \( p'(x) = 0 \)

   \[
   \begin{array}{c|cccccc}
   x & -\infty & -4 & 0 & 1 & +\infty \\
p(x) & +\infty & -9 & -1 & -4 & +\infty \\
   \end{array}
   \]

   \[ \therefore \] There is one real root with \( x < -4 \) and another with \( x > 1 \).

   There is no real root with \( -4 < x < 1 \).

   Since \( p(-5) < 0 \), \( p(-6) > 0 \) and \( p(1) < 0 \) and \( p(2) > 0 \).

   There are two real roots \( \alpha, \beta \) for \( p(x) = 0 \) with \( -6 < \alpha < -5 \) and \( 1 < \beta < 2 \).

   **(b)** Let \( p(x) = 8x^5 - 5x^4 - 40x^3 - 50 \), then \( p'(x) = 40x^4 - 20x^3 - 120x^2 = 20x^2(2x + 3)(x - 2) \)

   \[ x = -3/2, 0, 2 \] are roots of the equation \( p'(x) = 0 \)

   \[
   \begin{array}{c|cccccc}
   x & -\infty & -3/2 & 0 & 2 & +\infty \\
p(x) & -\infty & - & - & - & +\infty \\
   \end{array}
   \]

   \[ \therefore \] There is one real root with \( x > 2 \).

   Since \( p(3) < 0 \), \( p(4) > 0 \), there is one real root \( \alpha \) for \( p(x) = 0 \) with \( 3 < \alpha < 4 \).

2. **(a)** Let \( p(x) = x^4 + 2x^2 + 3x - 1 \).

   The number of sign change for \( p(x) \) is 1.

   By the Decartes' rule of sign, there is at most one positive root.

   \[ p(-x) = x^4 + 2x^2 - 3x - 1 \]

   The number of sign change for \( p(-x) \) is 1.

   By the Decartes' rule of sign, there is at most one negative root.

   There are at most two real roots.

   Since \( \deg[p(x)] = 4 \), there is at least 2 complex roots.

   \( p(-2) < 0 \), \( p(-1) < 0 \) and \( p(0) < 0 \), \( p(1) > 0 \).

   \[ \therefore \] There are 2 complex roots and 2 real roots \( \alpha, \beta \) for \( p(x) = 0 \) with \( -2 < \alpha < -1 \) and \( 0 < \beta < 1 \).

   **(b)** Let \( p(x) = x^5 - 2x^3 + x - 10 \), then \( p'(x) = 5x^4 - 6x^2 + 1 = (5x^2 - 1)(x^2 - 1) \)

   \[ x = -1, -\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 1 \] are roots of the equation \( p'(x) = 0 \)

   \[
   \begin{array}{c|cccccc}
   x & -\infty & -1 & -\frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 1 & +\infty \\
p(x) & -\infty & - & - & - & - & +\infty \\
   \end{array}
   \]

   \[ \therefore \] There is one real root with \( x > 1 \).

   Since \( p(1) < 0 \), \( p(2) > 0 \), there is one real root \( \alpha \) for \( p(x) = 0 \) with \( 1 < \alpha < 2 \).
3. \( \alpha, \beta, \gamma \) are the roots of the equation \( x^3 - px + q = 0 \) \( \ldots \) (1)
\[ \alpha + \beta + \gamma = 0 \] \( \ldots \) (2)
\[ \alpha\beta + \beta\gamma + \gamma\alpha = -p \] \( \ldots \) (3)
\[ \alpha\beta\gamma = -q \] \( \ldots \) (4)

Now, \( (\alpha - \beta)^2 = \alpha^2 + \beta^2 - 2\alpha\beta = \alpha^2 + \beta^2 + \gamma^2 - \frac{2\alpha\beta\gamma}{\gamma} = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) - \frac{\gamma^2}{\gamma} \)
\[ = 0^2 - 2(-p) - \gamma^2 - \frac{2(-q)}{\gamma} = 2p - \gamma^2 + \frac{2q}{\gamma} \], by (1) - (4).

\( (\alpha - \beta)^2 , (\beta - \gamma)^2 , (\gamma - \alpha)^2 \) are roots of a new equation formed by putting:
\[ y = 2p - x^2 + \frac{2q}{x} \]
\[ \Leftrightarrow xy = 2px - x^3 + 2q \Leftrightarrow (x^3 - px + q) + xy - px = 3q \Leftrightarrow xy - px = 3q \Leftrightarrow x = \frac{3q}{y-p} \] \( \ldots \) (5)

Substitute (5) in (1),
\[ \left( \frac{3q}{y-p} \right)^3 - p \left( \frac{3q}{y-p} \right) + q = 0 \Leftrightarrow y^3 - 6py^2 + 9p^2y + (27q^2 - 4p^3) = 0 \] \( \ldots \) (6)

The product of roots of (6) \[ D = (\alpha - \beta)^2 (\beta - \gamma)^2 (\gamma - \alpha)^2 = -(27q^2 - 4p^3) = 4p^3 - 27q^2 \] \( \ldots \) (7)

Since (1) is of degree 3, the condition of roots are shown below:
(i) 3 real and distinct roots \( \Leftrightarrow D > 0 \).
(ii) 1 real double root and one other real root or 1 triple root \( \Leftrightarrow D = 0 \).
(iii) 2 complex roots and 1 real roots \( \Leftrightarrow D < 0 \).

(w.l.o.g. let \( \alpha = a + bi \), \( \beta = a - bi \), \( \gamma = c \) and substitute in (7))

\( \therefore \) The roots of (1) should be real \( \Leftrightarrow D \geq 0 \) \( \Leftrightarrow 4p^3 - 27q^2 \geq 0 \).

4. \( x^2 - 3x + 4 = \lambda(1 + 2x) \) \( \Leftrightarrow x^2 - (3 + 2\lambda)x + (4 - \lambda) = 0 \)

This quadratic equation has real roots \( \Leftrightarrow \Delta = (3 + 2\lambda)^2 - 4(4 - \lambda) \geq 0 \)
\[ \Leftrightarrow 4\lambda^2 + 16\lambda \geq 0 \]
\[ \Leftrightarrow \lambda \leq \frac{-4 - \sqrt{21}}{2} \quad \text{or} \quad \lambda \geq \frac{4 + \sqrt{21}}{2} \]

\( x^2 - 3x + 4 = \lambda(1 + 2x) \)
\[ \Rightarrow \begin{cases} y = x^2 - 3x + 4 \quad \ldots \ldots \text{(1)} \\ y = \lambda(1 + 2x) \quad \ldots \ldots \text{(2)} \end{cases} \]

(2) is a pencil of straight lines passing through the point \( \left( -\frac{1}{2}, 0 \right) \).

(1) and (2) cut each other only when
\[ \lambda \leq \frac{-4 - \sqrt{21}}{2} \quad \text{or} \quad \lambda \geq \frac{4 + \sqrt{21}}{2} \]
as shown in the graph.

When \( \lambda = 3 \), (2) becomes \( y = 3 + 6x \).

When \( \frac{9 - \sqrt{77}}{2} < x < \frac{9 + \sqrt{77}}{2} \) \( \quad \text{(0.11 < x < 8.89)} \), \( 3 + 6x \) greater than \( (x^2 - 3x + 4) \).

5. Let \( y = (x - 1)^2 (x - a) + t \) \( \quad (a > 1) \)
\[ y' = (x - 1) (3x - 2a - 1) \quad , \quad y'' = 2(3x - a - 2) \]
For critical values, \( y' = 0 \).

\[ y' = 2(1 - a) < 0 \quad \text{since} \quad a > 1. \]

\[ x = 1 \quad \text{or} \quad x = \frac{2a + 1}{3}. \]

\[ y''_x = 2a + 1 = 2(a - 1) > 0 \quad \text{since} \quad a > 1. \]

\[ y \quad \text{is a min. when} \quad x = \frac{2a + 1}{3}, \quad y_{\text{min}} = -\frac{4(a - 1)^3}{27} + t \]

As \( x \to -\infty, \ y \to -\infty \) \quad \text{and} \quad \quad \text{as} \quad x \to +\infty, \ y \to +\infty.

The equation \( y = 0 \) has three real roots \( \iff \ y_{\text{max}} > 0 \land y_{\text{min}} < 0 \)

\[ t > 0 \quad \land \quad -\frac{4(a - 1)^3}{27} + t < 0 \quad \iff \quad 0 < t < \frac{4(a - 1)^3}{27} \]

\[ y = (x - 1)^2 (x - a) + t \quad \iff \quad y = x^3 - (2 + a)x^2 + (1 - 2a)x + (a + t) \]

\[ \alpha + \beta + \gamma = 2 + a \]

\[ \alpha \beta + \beta \gamma + \gamma \alpha = 1 - 2a \]

\[ \alpha \beta \gamma = a + t \]

\[ \beta \gamma + \gamma \alpha + \alpha \beta - 2(\alpha + \beta + \gamma) + 3 = 1 - 2a - 2(2 + a) + 3 = 0 \]

6. Let \( x = \frac{p}{q} \) be a rational root of \( ax^2 + bx + c = 0 \), where \( p, q \in \mathbb{Z}, \  \text{H.C.F} (p, q) = 1, \ p, q \neq 0 \).

By the rational zero theorem, \( p/c \quad \text{and} \quad q/a \).

Since \( a, c \) are odd and \( p, q \neq 0, \ p, q \) are odd.

However \( ax^2 + bx + c = 0 \quad \iff \quad a \left( \frac{p}{q} \right)^2 + b \left( \frac{p}{q} \right) + c = 0 \quad \Rightarrow \quad ap^2 + bpq + cq^2 = 0 \quad (q \neq 0) \)

There is a contradiction since \( ap^2, bpq, cq^2 \) are all odd so their sum is also odd, and must not be equal to \( 0 \).

7. (a) \( P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \)

\[ P(p) - P(q) = a_n \left( p^n - q^n \right) + a_{n-1} \left( p^{n-1} - q^{n-1} \right) + \ldots + a_1 (p - q) \]

\[ = (p - q) \left[ a_n (p^{n-1} + p^{n-2}q + \ldots + q^{n-1}) + a_{n-1} (p^{n-2} + p^{n-3}q + \ldots + q^{n-2}) + \ldots + a_1 \right] \]

Since \( a_0, a_1, \ldots, a_n, p, q \) are integers, the second factor in the above expression is an integer.

The first factor \( (p - q) \) is an even integer if \( p, q \) are both even integers (or both odd integers).

\( \therefore \ P(p) - P(q) \quad \text{is even} \).

(b) Suppose, on contrary, \( x = q \) is an integral root of \( P(x) = 0 \). Then \( P(q) = 0 \).

(i) If \( q \) is odd, then \( P(1) - P(q) \) is even by (a).

But \( P(1) - P(q) = P(1) - 0 = P(1) \), which is odd by given.

(ii) If \( q \) is even, then \( P(0) - P(q) \) is even by (a).

But \( P(0) - P(q) = P(0) - 0 = P(1) \), which is odd by given.

In both cases, there is a contradiction.
8. If $\alpha$ is the root of the equation $ax^2 + bx + c = 0$, then $a\alpha^2 + b\alpha + c = 0$ and $a\alpha^2 = -b\alpha - c$ .... (1)

$A\alpha^2 + B\alpha + C$ is rational $\iff a(A\alpha^2 + B\alpha + C) = A(-b\alpha - c) + B\alpha + C$, by (1)

$= (-Ab + Ba)\alpha - Ac + Ca$ is rational

$\Rightarrow -Ab + Ba = 0$, since $\alpha$ is irrational and all other constants are rational.

$\Rightarrow Ab = Ba$ .... (2)

From (1), $a\alpha^3 = -b\alpha^2 - c\alpha$ .... (3)

$A\alpha^3 + B\alpha^2 + C\alpha$ is rational $\iff A(a\alpha^3 + B\alpha^2 + C\alpha) = A(-b\alpha^2 - c\alpha) + Ba\alpha^2 + Ca\alpha$, by (2)

$= (Ba - Ab)\alpha^2 + (Ca - Ac)\alpha$ is rational, by (3)

$\Rightarrow Ca - Ac = 0$, since $\alpha$ is irrational and all other constants are rational.

$\Rightarrow Ac = Ca$

Let $p(x) = a_nx^n + a_{n-1}x^{n-1} + ... + a_0$. Then by Division Algorithm,

$p(x) = (ax^2 + bx + c)x + mx + n$, where $m, n$ are rational constants.

Now, $p(x)$ is rational $\iff (a\alpha^2 + b\alpha + c)q(\alpha) + m\alpha + n$ is rational .

$\Rightarrow m\alpha + n$ is rational , given that $a\alpha^2 + b\alpha + c = 0$

$\Rightarrow m = 0$, since $\alpha$ is irrational

$\Rightarrow p(x) = (ax^2 + bx + c)x + n$, that is, the remainder is independent of $x$.

9. Let $y = p(x) = x^3 - px + q$ then $p'(x) = 3x^2 - p$, $p''(x) = 6x$

Since $\lim_{x \to \infty} p(x) = +\infty$, $\lim_{x \to -\infty} p(x) = -\infty$, $p(x) = 0$ has at least one real root.

For stationary points, $p'(x) = 0$, $3x^2 - p = 0$ $\Rightarrow x = \pm \sqrt[3]{\frac{p}{3}}$ .... (1)

From (1), There are two turning points if $p > 0$.

Since $p''\left(\pm \sqrt[3]{\frac{p}{3}}\right) > 0$, $y$ is a min when $x = \pm \sqrt[3]{\frac{p}{3}}$, $y_{\min} = q - \frac{2p}{3}\sqrt[3]{\frac{p}{3}}$.

Since $p''\left(-\sqrt[3]{\frac{p}{3}}\right) > 0$, $y$ is a max when $x = -\sqrt[3]{\frac{p}{3}}$, $y_{\max} = q + \frac{2p}{3}\sqrt[3]{\frac{p}{3}}$.

$\therefore p(x) = 0$ has 3 real roots (or 1 double real root and 1 real root) $\iff y_{\min} \leq 0$ and $y_{\max} \geq 0$

$\Rightarrow q - \frac{2p}{3}\sqrt[3]{\frac{p}{3}} \leq 0$ and $q + \frac{2p}{3}\sqrt[3]{\frac{p}{3}} \geq 0$ $\iff \left(q - \frac{2p}{3}\sqrt[3]{\frac{p}{3}}\right)\left(q + \frac{2p}{3}\sqrt[3]{\frac{p}{3}}\right) \leq 0$

$\Rightarrow 4p^3 \geq 27q^2$

$y$ has only one real root $\iff [y_{\min} > 0$ and $y_{\max} > 0]$ or $[y_{\min} < 0$ and $y_{\max} < 0]$

$\Rightarrow \left(q - \frac{2p}{3}\sqrt[3]{\frac{p}{3}}\right)\left(q + \frac{2p}{3}\sqrt[3]{\frac{p}{3}}\right) > 0$ $\Rightarrow 4p^3 < 27q^2$.

(a) $x^3 - 2x + 7 = 0$, Since $4p^3 = 4(2)^3 = 32$, $27q^2 = 27(7)^2 = 1323$, $\therefore 4p^3 < 27q^2$.

The equation has only one real root.
(b) \(3x^3 + 4x - 2 = 0\). Since \(4p^3 = 4(-4/3)^3 = -256/27\), \(27q^2 = 27(-2/3)^2 = 12\), \(4p^3 < 27q^2\).

The equation has only one real root.

(c) \(4x^3 - 7x + 3 = 0\). Since \(4p^3 = 4(7/4)^3 = 343/16\), \(27q^2 = 27(3/4)^2 = 243/16\), \(4p^3 > 27q^2\).

The equation has 3 real roots.

10. (a) \(x^3 + Px^2 + Qx + R = 0\) \(\ldots (1)\)

Put \(x = X + k\), where \(k\) is a constant.

\((X + k)^3 + P(X + k)^2 + Q(X + k) + R = 0\) \(\ldots (2)\)

In (2), the coeff. of \(x^2\)-term is \((3k + P)\).

By putting \(3k + P = 0\) or \(k = -\frac{P}{3}\), then the transformation \(x = X - \frac{P}{3}\) can transform (1) into (2) where \(x^2\)-term disappears.

(b) \(x^3 - 15x = 126\) \(\ldots (3)\) \(x = y + z\) \(\ldots (4)\)

\((y + z)^3 - 15(y + z) = 126 \quad \Rightarrow \quad y^3 + z^3 + 3yz(y + z) - 15(y + z) = 126 \quad \Rightarrow \quad y^3 + z^3 + 3yz(y + z) = 126 \ldots (5)\)

If further we choose \(3yz - 15 = 0\), or \(yz = 5\) \(\ldots (6)\)

(5) becomes: \(y^3 + z^3 = 126\) \(\ldots (7)\)

From (6), \((y^3)(z^3) = 125\) \(\ldots (8)\)

From (7) and (8), \(y^3\) and \(z^3\) are roots of a quadratic equation \(t^2 - 126t + 125 = 0\) \(\ldots (9)\)

If \(y^3\) and \(z^3\) are both real and \(y_0, z_0\) are their real roots, possible values for \(y, z\) are \(y_0, y_0\omega, y_0\omega^2, z_0, z_0\omega, z_0\omega^2\).

By (6), the product \(yz\) must be real. Therefore the only combinations consistent with this give, as the three roots of the cubic, by (4): \(y_0 + z_0, \quad \omega y_0 + \omega^2 z_0, \quad \omega^2 y_0 + \omega z_0\).

Similarly for the eq. \(x^3 - 15x = 126\).

From (9), \(t = 1\) or \(125\). Take \(y_0 = 1\), \(z_0 = 5\), we have \(y_0 = 1, \quad z_0 = 5\).

Then the solutions are \(1 + 5, \quad \omega + 5\omega^2, \quad \omega^2 + 5\omega\).

11. (a) \(x^2 + px + q = 0 \quad \Rightarrow \quad \alpha + \beta = -p, \quad \alpha\beta = q \quad \ldots \) \(\ldots (1)\)

\((\alpha - \beta)(\beta - \alpha) = \alpha^3 - \beta^3 + \alpha\beta - (\alpha^3 + \beta^3) = (\alpha^3 - \beta^3) + \alpha\beta - [(\alpha + \beta)^3 - 4\alpha\beta(\alpha^2 + \beta^2) - 6\alpha\beta^2]\)

\(= (\alpha^3 - \beta^3) + \alpha\beta - [(\alpha + \beta)^3 - 4\alpha\beta(\alpha^2 + \beta^2) - 6\alpha\beta^2] = q^3 + q - [(\alpha + \beta)^3 - 4\alpha\beta(\alpha^2 + \beta^2) - 6\alpha\beta^2]\)

One of the roots of the equation \(x^2 + px + q = 0\) is equal to the cube of the other

\(\Rightarrow \quad \alpha^3 = \beta\) or \(\beta^3 = \alpha\quad \Rightarrow \quad (\alpha^3 - \beta^3)(\beta^3 - \alpha^3) = 0 \quad \Rightarrow \quad q(q + 1)^2 - (p^2 - 2q)^2 = 0\), by (2)

\(\Rightarrow \quad (p^2 - 2q)^2 = q(q + 1)^2 \ldots (2)\)

(b) \(x^2 + px + q = 0 \quad \Rightarrow \quad \alpha + \beta = -p, \quad \alpha\beta = q \quad \ldots \) \(\ldots (1)\)

\((\alpha + \beta)^2 = (\beta + \alpha)^2 = (1 + t)(\alpha + \beta) = (1 + t)(-p) = -p(1 + t)\)

\((\alpha + \beta)(\beta + \alpha) = (\alpha^2 + \beta^2) + \alpha\beta + \alpha^2\beta + \alpha^2\beta = t((\alpha + \beta)^2 - 2\alpha\beta) + \alpha\beta + \alpha\beta = t((\alpha + \beta)^2 - 2\alpha\beta) + q + t^2 q\)

\(= t^2 q + t(p^2 - 2q) + q\)

Hence the new equation is \(x^2 + p(1 + t)x + t^2 q + t(p^2 - 2q) + q = 0\)

If \(p^2 - 4q < 0\), \(p \neq 0\), then (1) has two complex conjugate roots \(\alpha\) and \(\beta\).

Since \(p, q\) are real, and \(\alpha\) and \(\beta\) are complex conjugates,
If \( t \neq 1 \) and is real, \((t\alpha + \beta), (t\beta + \alpha)\) are complex numbers.

(2) If \( t = 1 \), \((t\alpha + \beta), (t\beta + \alpha)\) are equal to \(-p\).

Since \( p \neq 0 \), therefore \((t\alpha + \beta), (t\beta + \alpha)\) are different from zero.

12. (a) \( p^2 \) and \( q \) are the roots of \( x^2 + bx + q = 0 \) \( \ldots (1) \)

\[ p^4 + p^2 b + q = 0 \] \( \ldots (2) \), \[ q^2 + b q + q = 0 \] \( \ldots (3) \)

Case (1), If \( q \neq 0 \), then from (3), \( q + b + 1 = 0 \) \( \Rightarrow q = -(b + 1) \) \( \ldots (4) \)

\[ (4) \downarrow (2), \] \[ p^4 + p^2 b + -(b + 1) = 0 \] \( \Rightarrow (p^2 + b + 1)(p^2 - 1) = 0 \)

\[ \therefore p = \pm \sqrt{-b + 1} \] or \( p = \pm 1 \).

Case (2), If \( q = 0 \), then (2) becomes \( p^4 + p^2 b = 0 \) \( \Rightarrow p^2 (p^2 + b) = 0 \)

\[ \therefore p = 0 \] or \( \pm \sqrt{-b} \)

(b) The equation \( x^2 + bx + c = 0 \) \( \Rightarrow \alpha + \beta = -b \) , \( \alpha \beta = c \) \( \ldots (1) \)

\[ \frac{1}{1+\alpha^2} + \frac{1}{1+\beta^2} = \frac{(\alpha + \beta)^2 + 2}{1 + (\alpha + \beta)(1 + \beta^2)} = \frac{\alpha^2 + \beta^2 + 2}{1 + (\alpha + \beta)(1 + \beta^2)} = \frac{b^2 - 2c + 2}{1 + b^2 + 2c + c^2} \]

\[ \therefore \text{The new equation is} \quad (1 + b^2 - 2c + c^2)X^2 + (2c - 2 - b^2)X + 1 = 0 \]

If \( b = 1 \) and \( c = -1 \), then the given equation is \( x^2 + x - 1 = 0 \), with roots \( x = \frac{-1 \pm \sqrt{5}}{2} \)

The new equation is \( 5X^2 - 5X + 1 = 0 \), with roots \( x = \frac{5 \pm \sqrt{5}}{10} \).

13. Let \( \frac{\alpha}{r}, \alpha, \alpha r \) be the roots of \( x^3 + 3x^2 + bx + c = 0 \) \( \ldots (1) \)

Then \( \alpha \left(\frac{1}{r} + 1 + r\right) = -3 \) \( \ldots (2) \), \( \alpha^2 \left(\frac{1}{r} + 1 + r\right) = b \) \( \ldots (3) \), \( \alpha^3 = -c \) \( \ldots (4) \)

(3)/(2), \( \alpha = -\frac{b}{3} \) \( \ldots (5) \), \( (5) \downarrow (4), \) \( c = \frac{b^2}{27} \) \( \ldots (6) \)

(6) \( \downarrow (1), \) \( x^3 + 3x^2 + bx + \frac{b^2}{27} = 0 \). From (5) and division, \( \left(x + \frac{b}{3}\right)\left(x^2 + \left(3 - \frac{b}{3}\right)x + \frac{b^2}{9}\right) = 0 \)

\[ \therefore x = -\frac{b}{3} \text{ or } x = \frac{-\left(3 - \frac{b}{3}\right) \pm \sqrt{\left(3 - \frac{b}{3}\right)^2 - 4 \cdot \frac{b^2}{9}}}{2} \ldots (7) \]

In (7), \( \Delta = \left(3 - \frac{b}{3}\right)^2 - 4 \cdot \frac{b^2}{9} = 9 - 2b + \frac{b^2}{3} - \frac{4b^2}{9} = \frac{27 - 6b - b^2}{3} \ldots (8) \)

From (5), Since \( \alpha \) is an integer and \( \alpha \neq 0 \), \( b \) must be a non-zero integer divisible by \( 3 \ldots (9) \)

In (8), Since the roots are integers, \( \Delta > 0 \)
\[ b^2 + 6b - 27 = (b + 9)(b - 3) \leq 0 \Rightarrow -9 < b < 3 \quad \ldots \quad (10) \]

The only \( b \) satisfying (9) and (10) is that \( b = -6 \) and \( \Delta = 9 \).

From (7), \( x = 2, -1, -4. \)

14. \( x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 = 0 \quad \ldots \quad (1) \)

\( z \) is a root of (1), \( z^2 + a_3 z^3 + a_2 z^2 + a_1 z + a_0 = 0 \quad \ldots \quad (2) \)

\( 1/z \) is a root of (1), \( (1/z)^2 + a_3 (1/z)^3 + a_2 (1/z)^2 + a_1 (1/z) + a_0 = 0 \)

\[ a_0 z^3 + a_1 z^2 + a_2 z + a_3 + 1 = 0 \quad \Rightarrow \quad z^4 + \left( \frac{a_1}{a_0} \right) z^3 + \left( \frac{a_2}{a_0} \right) z^2 + \left( \frac{a_3}{a_0} \right) z + \left( \frac{1}{a_0} \right) = 0 \quad \ldots \quad (3) \]

Since \( z \) and \( 1/z \) are roots, (2) and (3) are identical.

By comparing coefficients of constant terms of (2) and (3), \( a_0 = \frac{1}{a_0} \Rightarrow a_0^2 = 1 \Rightarrow a_0 = \pm 1 \)

Case (1), When \( a_0 = 1, \ a_1 = a_3 \) (by comparing coeffs of \( x^2 \)-terms)

Case (2), When \( a_0 = -1, \ a_1 = -a_3, \ a_2 = 0 \) (by comparing coeffs of \( x^3 \)-terms and \( x^2 \)-terms)

\( \therefore \) The necessary and sufficient conditions are: \[ [(a_0 = 1, \ a_1 = a_3) \text{ or } (a_0 = -1, \ a_1 = -a_3, \ a_2 = 0)] \]

Put \( p(x) = x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \)

For Case (2), \[ p(1) = 1 + a_3 + a_2 + a_1 + a_0 = 0, \quad p(-1) = 1 - a_3 + a_2 - a_1 + a_0 = 0. \]

\( \therefore \) \((x - 1)(x + 1)\) is a factor of \( p(x) \) by Factor Theorem.

Division of \( p(x) \) by \((x - 1)(x + 1)\) gives a quadratic factor and the problem reduces to solving quadratic equation.

For Case (1), equation (1) reduces to \( x^4 + a_1 x^3 + a_2 x^2 + a_1 x + 1 = 0 \)

\[ \Rightarrow x^2 + a_1 x + a_2 + \frac{a_1}{x} + \frac{1}{x^2} = 0 \Rightarrow \left( x + \frac{a_1}{x} + \frac{1}{x^2} \right) + a_2 = 0 \quad \ldots \quad (4) \]

Put \( y = x + \frac{1}{x} \) \quad \ldots \quad (5)

Then \( x^2 + \frac{1}{x^2} = \left( x + \frac{1}{x} \right)^2 - 2 = y^2 - 2 \)

(4) becomes \( y^2 - 2 + a_1 y + a_2 = 0 \) or \( y^2 - 2 + a_2 y + (a_2 - 2) = 0 \) \quad \ldots \quad (6)

(6) is a quadratic and let \( y_1 \) and \( y_2 \) be the roots. From (5), we have:

\[ x^2 - y_1 x + 1 = 0 \quad \text{and} \quad x^2 - y_2 x + 1 = 0 \quad \ldots \quad (7) \]

which are quadratic equations in \( x \) and (7) gives the roots of (1).

15. \( x^4 - s_1 x^3 + s_2 x^2 - s_3 x + s_4 = 0 \quad \ldots \quad (1) \)

Writing \( \Sigma \) as symmetric sum, then

\[ \Sigma a = s_1 \quad \ldots \quad (2) \quad , \quad \Sigma a b = s_2 \quad \ldots \quad (3) \quad , \quad \Sigma a b \gamma = s_3 \quad \ldots \quad (4) \quad , \quad a b \gamma \delta = s_4 \quad \ldots \quad (5) \]

\[ \Sigma (a b + \gamma \delta) = \Sigma a b = s_2 \quad \ldots \quad (6) \]

\[ \Sigma (a b + \gamma \delta)(a \gamma + b \delta) = \Sigma a^2 b \gamma = (\Sigma a)(\Sigma a b \gamma) - 4a b \gamma \delta = s_1 s_3 - 4 s_4 \quad \ldots \quad (7) \]

\[ (a b + \gamma \delta)(a \gamma + b \delta)(a \delta + b \gamma) = \Sigma a^2 b^2 \gamma^2 = (a b \gamma \delta + \Sigma a^2 b \gamma^2) \Sigma a \gamma \delta + (a b \gamma \delta) \Sigma a^2 b \gamma^2 + [(\Sigma a b \gamma)^2 - 2 \Sigma a^2 b \gamma^2 \gamma \delta] \]

\[ = (a b \gamma \delta)(\Sigma a^2) - 2 \Sigma a b ] +[(\Sigma a b \gamma)^2 - 2a b \gamma \delta \Sigma a b \gamma] = s_4 [s_1^2 - 2s_2] + [s_3^2 - 2s_4 s_2] = -4s_2 s_4 - s_1^2 s_4 - s_3^2 \ldots \quad (8) \]

From (6), (7), (8), the required cubic equation is
\[ y^3 - s_2 y^2 + (s_1 s_3 - 4 s_4)y + (4s_2 s_4 - s_1 s_3^2 - s_4^2) = 0 \quad \ldots \quad (9) \]

Supposing that methods of solving cubic equation are known, (9) is solved and let the roots be \( y_1, y_2 \) and \( y_3 \). Then:

\[ \alpha \beta + \gamma \delta = y_1 \quad \ldots \quad (10) \, , \quad \alpha \gamma + \beta \delta = y_2 \quad \ldots \quad (11) \, , \quad \alpha \delta + \beta \gamma = y_3 \quad \ldots \quad (12) \]

From (5), \( (\alpha \beta)(\gamma \delta) = s_4 \quad \ldots \quad (13) \)

From (10) and (13), \( \alpha \beta \) and \( \gamma \delta \) are roots of the quadratic equation:

\[ z^2 - y_1 z + s_4 = 0 \quad \ldots \quad (14) \]

Suppose (14) is solved, let the roots be \( z_1 = \alpha \beta \, , \, z_2 = \gamma \delta \). \( \ldots \quad (15) \)

From (3), \( \gamma \delta (\alpha + \beta) + \alpha \beta (\gamma + \delta) = s_3 \quad \Leftrightarrow \quad z_2 (\alpha + \beta) + z_1 (\gamma + \delta) = s_3 \quad \ldots \quad (16) \)

From (2), \( (\alpha + \beta) + (\gamma + \delta) = s_1 \quad \ldots \quad (17) \)

(17) and (18) are simultaneous equation with unknowns \( \alpha, \beta, \gamma, \delta \).

Let the equations be solved and \( k_1 = \alpha + \beta \, , \, k_2 = \gamma + \delta \) \( \ldots \quad (18) \)

(15), (16) form two quadratic equations for which \( \alpha, \beta, \gamma, \delta \) can be found.

Note that if (11) or (12) is used instead of (10), the solution is the same due to the arbitrary use of the notation \( \alpha, \beta, \gamma, \delta \) for the four roots of (1).

16. (a) We use induction on \( k \) that \( x^k + x^{-k} \) can be expressed as a polynomial \( p_k(z) \) in \( z = x + x^{-1} \).

The proposition is obviously true for \( k = 1 \) where \( p_1(z) = z \).

For \( k = 2 \), \( x^2 + x^{-2} = (x + x^{-1})^2 - 1 = z^2 - 2 \) and the proposition is true for \( k = 2 \).

Assume the proposition is true for \( k \leq n \) where \( n \in \mathbb{N} \).

For \( k = n + 1 \), \( x^{n+1} + x^{-n-1} = (x^n + x^{-n})(x + x^{-1}) - [x^{n-1} + x^{-n-1}] \)

\[ = p_n(z) p_1(z) - p_{n-1}(z) \quad , \text{by inductive hypothesis.} \]

Putting \( p_{n+1}(z) = p_n(z) p_1(z) - p_{n-1}(z) \), then the proposition is also true for \( k = n + 1 \).

By the Principle of Mathematical Induction, the proposition is true \( \forall k \in \mathbb{N} \).

(b) \( x^6 + ax^5 + bx^4 + cx^3 + bx^2 + ax + 1 = 0 \quad \ldots \quad (1) \)

\( (x^3 + x^2) + a(x^2 + x) + b(x + x^{-1}) + c = 0 \quad \ldots \quad (2) \)

Putting \( z = x + x^{-1} \), then (1) becomes \( (z^3 - 3z) + a+ bz + c = 0 \).

or \( z^3 = az^2 + (b - 3)z + (c - 2a) = 0 \quad \ldots \quad (3) \)

\( \therefore \) If \( \alpha \) is a root of the polynomial equation (1), then \( \alpha + \alpha^{-1} \) is a root of (2).

(c) \( x^4 + 4x^3 + 5x^2 + 4x + 1 = 0 \Leftrightarrow (x^2 + x^3) + 4( x + x^{-1}) + 5 = 0 \quad \ldots \quad (4) \)

Putting \( z = x + x^{-1} \), then (4) becomes \((z^2 - 2) + 4z + 5 = 0 \)

or \( z^2 + 4z + 3 = 0 \Leftrightarrow (z + 3)(z + 1) = 0 \)

\( z = x + x^{-1} = -3 \) or \( z = x + x^{-1} = -1 \)

\( x^2 + 3x + 1 = 0 \) or \( x^2 + x + 1 = 0 \)

\( x = \frac{-3 \pm \sqrt{5}}{2} \) or \( x = \frac{-1 \pm \sqrt{3}i}{2} \)
17. \( x^3 + 3qx + r = 0 \) \( (r \neq 0) \) \ldots (1)

\( \alpha + \beta + \gamma = 0 \) \ldots (2) \quad \alpha \beta + \beta \gamma + \gamma \alpha = 3q \) \ldots (3) \quad \alpha \beta \gamma = -r \) \ldots (4)

Put \( p(x) = r^2(x^2 + x + 1)^3 + 27q^3x^2(x + 1)^2 \)

Then

\[
p\left(\frac{\alpha}{\beta}\right) = r^2\left(\frac{\alpha}{\beta^2} + \frac{\alpha}{\beta} + 1\right)^3 + 27q^3\frac{\alpha^2}{\beta^2} \left(\frac{\alpha}{\beta} + 1\right)^2 = r^2\left(\frac{\alpha^2 + \alpha + 1}{\beta^2}\right)^3 + 27q^3\frac{\alpha^2}{\beta^2} \left(\frac{\alpha + \beta}{\beta}\right)^3 \]

\[
= r^2\left(\frac{\alpha^3 - \beta^3}{\beta^3(\alpha - \beta)}\right)^3 + 27q^3\frac{\alpha^2}{\beta^3}(\gamma)^2 = r^2\left(\frac{-3q\alpha + r}{\beta^3(\alpha - \beta)}\right)^3 + 27q^3\frac{\alpha^2}{\beta^5} \frac{\gamma^2}{\beta^4} \]

\[
= r^2\left(\frac{-3q}{\beta^3}\right)^3 + 27q^3\frac{(\alpha \beta \gamma)^2}{\beta^5} = -\frac{27q^3r^2}{\beta^3} + 27q^3\frac{(-r)}{\beta^3} = 0
\]

\[
\Rightarrow \frac{\alpha}{\beta} \text{ satisfies } p(x) = r^2(x^2 + x + 1)^3 + 27q^3x^2(x + 1)^2 = 0 \quad \ldots (5)
\]

(a) When \( q = 0 \), (1) becomes \( x^3 + r = 0 \) \ldots (6)

The roots of (6) are \( \omega, \omega^2, \omega^3 \), where \( \omega \) is the complex cube roots of unity.

Take \( \alpha \) be any one of these three roots and \( \beta \) be any other root not equal to \( \alpha \).

Then \( \alpha = \omega \beta \text{ or } \alpha = \omega^2 \beta \) \ldots (7)

When \( q = 0 \), (5) becomes \( x^3 + x + 1 = 0 \) \( (r \neq 0) \) and the roots are \( \omega, \omega^2 \).

\[
\Rightarrow \frac{\alpha}{\beta} = \omega \text{ or } \frac{\alpha}{\beta} = \omega^2 \quad \ldots (8)
\]

It can be seen that (7) and (8) are the same.

(b) If \( 4q^3 + r^2 = 0 \), then \( r^2 = -4q^3 \) \ldots (9)

(9) \( \downarrow \) (5), \( p(x) = -4q^3(x^2 + x + 1)^3 + 27q^3x^2(x + 1)^2 = 0 \) \ldots (10)

If \( q = 0 \), then by (9), \( r = 0 \), contradicting to \( r \neq 0 \).

\[
\Rightarrow q \neq 0 \text{ and (10) becomes } -4(x^3 + x + 1)^3 + 27x^3(x + 1)^2 = 0
\]

\( \Rightarrow 4x^6 + 12x^5 - 3x^4 - 26x^3 - 3x^2 + 12x + 4 = 0 \)

\( \Rightarrow (x - 1)^2 (x + 2)^2 (2x + 1)^2 = 0 \)

Since \( \frac{\alpha}{\beta} \) satisfies (10), \( \frac{\alpha}{\beta} = 1 \) or \(-2\) or \(-\frac{1}{2}\).

(i) If \( \frac{\alpha}{\beta} = 1 \), then \( \alpha = \beta \).

(ii) If \( \frac{\alpha}{\beta} = -2 \), then \( \alpha = -2\beta \). From (1), \(-2\beta + \beta + \gamma = 0 \), \( \beta = \gamma \).

(iii) If \( \frac{\alpha}{\beta} = -\frac{1}{2} \), then \( \beta = -2\alpha \). From (1), \( -2\alpha + \gamma = 0 \), \( \alpha = \gamma \).

In any one of the above cases, there is a double root for (1).
18. (a) A.M. – G.M. : Let \( x_i \) (i = 1, 2, ..., n) be n distinct positive numbers, then
\[
\frac{\sum x_i}{n} > \Pi x_i.
\]

(b) \( f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_nx + a_n = 0 \) has n distinct positive roots \( \alpha_i \) (i = 1, 2, ..., n)
Then \( \sum \alpha_i = -a_1 \quad \ldots \quad (1) \), \( \sum \alpha_i \alpha_j = a_2 \quad \ldots \quad (2) \), \ldots,
\[
\sum \alpha_i \alpha_j \alpha_k = (-1)^{i+j+k} a_{i+j+k} \quad \ldots \quad (n-1), \quad \alpha_i \alpha_j \alpha_k = (-1)^i a_i \quad \ldots \quad (n)
\]
Writing \( a_i = (-1)^i \left( \frac{n}{i} \right) b_i \), then \( a_i = (-1)^i \left( \frac{n}{i} \right) b_i = (-1)^i b_i \quad \ldots \quad (i) \)
\[
a_i = (-1)^i \left( \frac{n}{i-1} \right) b_i = (-1)^i b_i \quad \ldots \quad (ii)
\]
From (i) and (n), \( b_i = \alpha_i \alpha_2 \alpha_3 \ldots \alpha_{n-i} \quad \ldots \quad (iii) \)
From (ii) and (n-1), \( nb_i = \sum_{n}^{\alpha_i \alpha_2 \ldots \alpha_{n-1}} \quad \ldots \quad (iv) \)

From (iii),
\[
b_i = \frac{\left[ \alpha_i \alpha_2 \ldots \alpha_{n-i} \right]^i}{\Pi} = \left[ \frac{\Pi \alpha_i \alpha_2 \ldots \alpha_{n-i}}{\Pi} \right]^i = \left[ \frac{\Pi \alpha_i \alpha_2 \ldots \alpha_{n-i}}{b_i} \right]^i
\]
\[
< \left[ \frac{\sum \alpha_i \alpha_2 \ldots \alpha_{n-i}}{n} \right]^i < \left[ \frac{b_i}{\Pi \alpha_{n-i}} \right]^i = b_i \quad \ldots \quad (v)
\]
\[
f'(x) = nx^{n-1} + (n-1)x^{n-2} + ... + 2a_{n-2}x + a_n = n\left[ x^{n-1} + \left( \frac{n-1}{n} \right)x^{n-2} + ... + \frac{2}{n}a_{n-2}x + \frac{1}{n}a_n \right]
\]
Consider \( f'(x) = 0 \).
Since \( f(x) = 0 \) has n distinct positive real roots and between any two roots of \( f(x) = 0 \), there is at least one real root of \( f'(x) = 0 \), where \( \deg f(x) = n-1 \).
\[
\therefore \quad f'(x) = 0 \quad \text{has} \quad (n-1) \quad \text{distinct positive real roots, each lying between two roots of} \quad f(x) = 0.
\]
Let \( \beta_1, \beta_2, \ldots, \beta_{n-1} \) be the (n-1) distinct positive real roots of \( f'(x) = 0 \)
\[
\sum \beta_i \beta_2 \ldots \beta_{n-2} = (-1)^{n-1} \frac{2}{n} a_{n-2} = (-1)^{n-2} \frac{2}{n} n \left[ (-1)^{n-2} \left( \frac{n}{n-2} \right) b_i \right] = \frac{2}{n} \frac{n(n-1)}{2} b_i = (n-1)b_{n-2} \quad \ldots \quad (vi)
\]
\[
\beta, \beta_2, \ldots \beta_{n-2} = (-1)^{n-1} a_{n-1} = (-1)^{n-1} \frac{1}{n} n \left[ (-1)^{n-1} nb_i \right] = b_{n-1} \quad \ldots \quad (vii)
\]
\[
b_i = \left( \beta, \beta_2, \ldots \beta_{n-2} \right)^{i-1} = \left[ \frac{\Pi \beta, \beta_2, \ldots \beta_{n-2}}{\Pi} \right]^{i-1} = \left[ \frac{\Pi \beta, \beta_2, \ldots \beta_{n-2}}{b_i} \right]^{i-1}
\]
\[
< \left[ \frac{\sum \beta, \beta_2, \ldots \beta_{n-2}}{n-1} \right]^{i-1} < \left[ \frac{(n-1)b_{n-2}}{n-1} \right]^{i-1} = b_{n-2} \quad \ldots \quad (viii)
\]
Similarly, by considering \( f''(x), f'''(x), \ldots, f^{(n-2)}(x) \), we get correspondingly,
\( b_{n-2} > b_{n-3}, \quad b_{n-3} > b_{n-4}, \quad \ldots, \quad b_2 > b_1 \)
Combining all these inequalities, we have \( b_1 > b_2 > \ldots > b_{n-1} > b_n \).