Rearrangement Inequality

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The rearrangement inequality (also known as permutation inequality) is easy to understand and yet a powerful tool to handle inequality problems.

Definition Let $a_1 \le a_2 \le \ldots \le a_n$ and $b_1 \le b_2 \le \ldots \le b_n$ be any real numbers.

(a) $S = a_1b_1 + a_2b_2 + \ldots + a_nb_n$ is called the ***Sorted sum*** of the numbers.

(b) $R = a_1b_n + a_2b_{n-1} + \ldots + a_nb_1$ is called the ***Reversed sum*** of the numbers.

(c) Let $c_1, c_2, \ldots, c_n$ be any permutation of the numbers $b_1, b_2, \ldots, b_n$.
$P = a_1c_1 + a_2c_2 + \ldots + a_nc_n$ is called the ***Permutated sum*** of the numbers.

Rearrangement inequality $S \ge P \ge R$

Proof

(a) Let P(n) be the proposition : $S \ge P$.
P(1) is obviously true.
Assume P(k) is true for some $k \in \mathbf{N}$.
For P(k + 1), Since the c's are the permutations of the b's, suppose $b_{k+1} = c_i$ and $c_{k+1} = b_j$

$$(a_{k+1} - a_i)(b_{k+1} - b_j) \ge 0 \quad \Rightarrow \quad a_ib_j + a_{k+1}b_{k+1} \ge a_i b_{k+1} + a_{k+1}b_j$$

$$\Rightarrow \quad a_ib_j + a_{k+1}b_{k+1} \ge a_i c_i + a_{k+1} c_{k+1}$$

So in P , we may switch c_i and c_{k+1} to get a possibly larger sum.
After switching of these terms, we come up with the inductive hypothesis P(k).
$\therefore$ P(k + 1) is also true.
By the principle of mathematical induction, P(n) is true $\forall n \in \mathbf{N}$.

(b) The inequality $P \ge R$ follows easily from $S \ge P$ by replacing $b_1 \le b_2 \le \ldots \le b_n$ by $-b_n \ge -b_{n-1} \ge \ldots \ge -b_1$.

Note:

(a) If a_i's are strictly increasing, then equality holds (S = P = R) if and only if the b_i's are all equal.

(b) Unlike most inequalities, we do not require the numbers involved to be positive.

Corollary 1 Let $a_1, a_2, \ldots, a_n$ be real numbers and $c_1, c_2, \ldots, c_n$ be its permuation. Then

$$a_1^2 + a_2^2 + \ldots + a_n^2 \ge a_1c_1 + a_2c_2 + \ldots + a_nc_n$$

Corollary 2 Let $a_1, a_2, \ldots, a_n$ be **positive** real numbers and $c_1, c_2, \ldots, c_n$ be its permuation. Then

$$\frac{c_1}{a_1} + \frac{c_2}{a_2} + \ldots + \frac{c_n}{a_n} \ge n$$

The rearrangement inequality can be used to prove many famous inequalities. Here are some of the highlights.

Arithmetic Mean - Geometric Mean Inequality (A.M. ≥ G.M.)

Let $x_1, x_2, \ldots, x_n$ be positive numbers. Then $\frac{x_1 + x_2 + \ldots + x_n}{n} \ge \sqrt[n]{x_1 x_2 \ldots x_n}$.

Equality holds if and only if $x_1 = x_2 = \ldots = x_n$.

Proof Let $G = \sqrt[n]{x_1 x_2 \ldots x_n}$, $a_1 = \frac{x_1}{G}, \ a_2 = \frac{x_1 x_2}{G^2}, \ \ldots, \ a_n = \frac{x_1 x_2 \ldots x_n}{G^n} = 1$.

By corollary 2, $n \le \frac{a_1}{a_n} + \frac{a_2}{a_1} + \ldots + \frac{a_n}{a_{n-1}} = \frac{x_1}{G} + \frac{x_2}{G} + \ldots + \frac{x_1}{G} \Leftrightarrow \frac{x_1 + x_2 + \ldots + x_n}{n} \ge \sqrt[n]{x_1 x_2 \ldots x_n}$

Equality holds $\Leftrightarrow$ $a_1 = a_2 = \ldots = a_n$ $\Leftrightarrow$ $x_1 = x_2 = \ldots = x_n$.

Geometric Mean –Harmonic Mean Inequality (G.M. ≥ H.M.)

Let $x_1, x_2, \ldots, x_n$ be positive numbers. Then $\sqrt[n]{x_1 x_2 \ldots x_n} \ge \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}}$

Proof Define G and $a_1, a_2, \ldots, a_n$ similarly as in the proof of A.M. – G.M.

By Corollary 2, $n \le \frac{a_1}{a_2} + \frac{a_2}{a_3} + \ldots + \frac{a_n}{a_1} = \frac{G}{x_1} + \frac{G}{x_2} + \ldots + \frac{G}{x_n}$ which then gives the result.

Root Mean Square - Arithmetic Mean Inequality (R.M.S. ≥ A.M.)

Let $x_1, x_2, \ldots, x_n$ be numbers. Then $\sqrt{\frac{x_1^2 + x_2^2 + \ldots + x_n^2}{n}} \ge \frac{x_1 + x_2 + \ldots + x_n}{n}$

Proof By Corollary 1, we cyclically rotate x_i ,

$$x_1^2 + x_2^2 + \ldots + x_n^2 = x_1x_1 + x_2x_2 + \ldots + x_nx_n$$
$$x_1^2 + x_2^2 + \ldots + x_n^2 \ge x_1x_2 + x_2 x_3 + \ldots + x_nx_1$$
$$x_1^2 + x_2^2 + \ldots + x_n^2 \ge x_1x_3 + x_2 x_4 + \ldots + x_nx_2$$
$$\ldots . \ge \ldots .$$
$$x_1^2 + x_2^2 + \ldots + x_n^2 \ge x_1x_n + x_2 x_1 + \ldots + x_nx_{n-1}$$

Adding all inequalities together, we have

$$n(x_1^2 + x_2^2 + \ldots + x_n^2) \ge (x_1 + x_2 + \ldots + x_n)^2$$

Result follows. Equality holds $\Leftrightarrow$ $x_1 = x_2 = \ldots = x_n$

Cauchy –Bunyakovskii – Schwarz inequality (CBS inequality)

Let $a_1, a_2, \ldots, a_n$; $b_1, b_2, \ldots, b_n$ be real numbers.

Then $(a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2 \le \left(a_1^2 + a_2^2 + \ldots + a_n^2\right)\left(b_1^2 + b_2^2 + \ldots + b_n^2\right)$

Proof The result is trivial if $a_1 = a_2 = \ldots = a_n = 0$ or $b_1 = b_2 = \ldots = b_n = 0$. Otherwise, define

$A = \sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}$, $B = \sqrt{b_1^2 + b_2^2 + \ldots + b_n^2}$

Since both A and B are non-zero, we may let $x_i = \frac{a_i}{A}, \ x_{n+i} = \frac{b_i}{B}$ $\forall\ 1 \le i \le n$.

By Corollary 1, $$2 = \frac{a_1^2 + a_2^2 + ... + a_n^2}{A^2} + \frac{b_1^2 + b_2^2 + ... + b_n^2}{B^2} = x_1^2 + x_2^2 + ... + x_{2n}^2$$

$$\geq x_1 x_{n+1} + x_2 x_{n+2} + ... + x_n + x_{2n} + x_{n+1} x_1 + x_{n+2} x_2 + ... + x_{2n} x_n$$

$$= \frac{2(a_1 b_1 + a_2 b_2 + ... + a_n b_n)}{AB}$$

$$\Leftrightarrow \quad (a_1 b_1 + a_2 b_2 + ... + a_n b_n)^2 \leq (a_1^2 + a_2^2 + ... + a_n^2)(b_1^2 + b_2^2 + ... + b_n^2)$$

Equality holds $\Leftrightarrow$ $x_i = x_{n+i}$ $\Leftrightarrow$ $a_i B = b_i A$ $\forall\ 1 \leq i \leq n$.

Chebyshev's inequality

Let $x_1 \leq x_2 \leq \ldots \leq x_n$ and $y_1 \leq y_2 \leq \ldots \leq y_n$ be any real numbers.

Then $$x_1 y_1 + x_2 y_2 + ... + x_n y_n \geq \frac{(x_1 + x_2 + ... + x_n)(y_1 + y_2 + ... + y_n)}{n} \geq x_1 y_n + x_2 y_{n-1} + ... + x_n y_1$$

Proof By Rearrangement inequality, we cyclically rotate x_i and y_i,

$$\begin{array}{ccccc} x_1y_1 + x_2y_2 + \ldots + x_ny_n & = & x_1y_1 + x_2y_2 + \ldots + x_ny_n & \geq & x_1y_n + x_2y_{n-1} + \ldots + x_ny_1 \\ x_1y_1 + x_2y_2 + \ldots + x_ny_n & \geq & x_1y_2 + x_2y_3 + \ldots + x_ny_1 & \geq & x_1y_n + x_2y_{n-1} + \ldots + x_ny_1 \\ \ldots & \geq & \ldots & \geq & \ldots \\ x_1y_1 + x_2y_2 + \ldots + x_ny_n & \geq & x_1y_n + x_2y_{n-1} + \ldots + x_ny_1 & = & x_1y_n + x_2y_{n-1} + \ldots + x_ny_1 \end{array}$$

Adding up the inequalities and divide by n, we get our result.

Exercise	Hint
1. Find the minimum of $\frac{\sin^3 x}{\cos x} + \frac{\cos^3 x}{\sin x},\ 0 < x < \frac{\pi}{2}$	Consider $(\sin^3 x, \cos^3 x)$, $\left(\frac{1}{\sin x}, \frac{1}{\cos x}\right)$
2. Proof : (i) $a^2 + b^2 + c^2 \geq ab + bc + ca$ (ii) $a^n + b^n + c^n \geq ab^{n-1} + bc^{n-1} + ca^{n-1}$	For (ii) and questions below, Without lost of generality, let $a \leq b \leq c$ Consider (a, b, c), $(a^{n-1}, b^{n-1}, c^{n-1})$
3. Proof : $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq \frac{a+b+c}{abc}$	Consider $\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$, $\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$
4. Proof : $\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \geq \frac{b}{a} + \frac{c}{b} + \frac{a}{c}$	Consider $\left(\frac{a}{b}, \frac{b}{c}, \frac{c}{a}\right)$, $\left(\frac{a}{b}, \frac{b}{c}, \frac{c}{a}\right)$
5. Proof : $\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq a + b + c$	Consider (a^2, b^2, c^2), $\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$
6. Proof : If $a, b, c > 0$ and $n \in N$ then $\frac{a^n}{b+c} + \frac{b^n}{c+a} + \frac{c^n}{a+b} \geq \frac{a^{n-1} + b^{n-1} + c^{n-1}}{2}$	Consider (a^n, b^n, c^n), $\left(\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}\right)$
7. Proof: If $a, b, c > 0$, then $a^a b^b c^c \geq (abc)^{\frac{a+b+c}{3}}$	Consider (a, b, c) , (log a, log b, log c) and use Chebyshev's inequality