Factorization of Cyclic and Symmetric polynomials

1. Definitions

(a) \( y = f(x, y, z) \) is **cyclic** if \( f(x, y, z) = f(y, z, x) \)

(b) A function in any number of variables is **symmetric** when it is unaltered by interchanging any two of the variables.

(c) A polynomial is **homogeneous** if the degree of each term is the same.

   e.g. \( f(x, y, z) = x^2y + y^2z + 2xyz \) is homogeneous, but \( g(x, y, z) = x^2yz + x \) is not.

2. Table of cyclic and symmetric polynomials:

<table>
<thead>
<tr>
<th>Degree</th>
<th>Cyclic</th>
<th>Symmetric</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( A(x + y + z) )</td>
<td>( A(x + y + z) )</td>
</tr>
<tr>
<td>2</td>
<td>( A(x^2 + y^2 + z^2) + B(xy +yz + zx) )</td>
<td>( A(x^2 + y^2 + z^2) + B(xy +yz + zx) )</td>
</tr>
<tr>
<td>3</td>
<td>( A(x^3 + y^3 + z^3) + B(x^2y + y^2z + z^2x) + C(xy^2 + yz^2 + zx^2) + Dxyz )</td>
<td>( A(x^3 + y^3 + z^3) + B(x^2y + y^2z + z^2x + xy^2 + yz^2 + zx^2) + Cxyz )</td>
</tr>
</tbody>
</table>

3. Useful Theorems

(a) The algebraic sum, difference, product and quotient of two cyclic (or symmetric) functions are cyclic (symmetric).

(b) **Factor theorem**

   \((x – y)\) is a factor of \( f(x, y, z) \) \iff \( f(y, y, z) = 0 \)

(c) A symmetric polynomial is cyclic but not vice versa.

   For example, \( f(x, y, z) = x^2y + y^2z + z^2x \) is cyclic but not symmetric.
4. An example

Factorize \( f(x, y, z) = x^3 (y - z) + y^3 (z - x) + z^3 (x - y) \)

\[
\begin{align*}
\text{Check that whether } f \text{ is cyclic / symmetric:} \\
f \text{ is cyclic but not symmetric.} \\
f \text{ is homogeneous of degree 4.}
\end{align*}
\]

\[
\begin{align*}
\text{ Sometimes, there is no need to expand all terms.} \\
\text{ Employ } (y - z)^{2n-1} = -(z - y)^{2n-1}
\end{align*}
\]

5. Some common factors of cyclic polynomial \( f(x, y, z) \):

<table>
<thead>
<tr>
<th>Factor</th>
<th>Test</th>
</tr>
</thead>
<tbody>
<tr>
<td>((x + y)(y + z)(z + x))</td>
<td>(f(-y, y, z) = 0)</td>
</tr>
<tr>
<td>((x - y)(y - z)(z - x))</td>
<td>(f(y, y, z) = 0)</td>
</tr>
<tr>
<td>(xyz)</td>
<td>(f(0, y, z) = 0)</td>
</tr>
<tr>
<td>(x + y + z)</td>
<td>(f(-y -z, y, z) = 0)</td>
</tr>
<tr>
<td>((x + y - z)((y + z - x)(z + x - y)))</td>
<td>(f(z - y, y, z) = 0)</td>
</tr>
</tbody>
</table>