Different kinds of Mathematical Induction

(1) Mathematical Induction

Given $A \subset \mathbb{N}$, $[1 \in A \land (a \in A \Rightarrow a+1 \in A)] \Rightarrow A = \mathbb{N}$.

(2) (First) Principle of Mathematical Induction

Let $P(x)$ be a proposition (open sentence), if we put $A = \{x : x \in \mathbb{N} \land p(x) \text{ is true}\}$ in (1), we get the Principle of Mathematical Induction.

If
   (1) $P(1)$ is true;
   (2) $P(k)$ is true for some $k \in \mathbb{N} \Rightarrow P(k+1)$ is true
then $P(n)$ is true $\forall n \in \mathbb{N}$.

(3) Second Principle of Mathematical Induction

If
   (1) $P(1)$ is true;
   (2) $\forall 1 \leq i \leq k, \ P(i)$ is true \[i.e. P(1) \land P(2) \land \ldots \land P(k) \text{ is true}] \Rightarrow P(k+1)$ is true
then $P(n)$ is true $\forall n \in \mathbb{N}$.

(4) Second Principle of Mathematical Induction (variation)

If
   (1) $P(1) \land P(2)$ is true;
   (2) $P(k-1) \land P(k)$ is true for some $k \in \mathbb{N}\{1\} \Rightarrow P(k+1)$ is true
then $P(n)$ is true $\forall n \in \mathbb{N}$.

(5) Second Principle of Mathematical Induction (variation)

If
   (1) $P(1) \land P(2) \land \ldots \land P(m)$ is true;
   (2) $P(k)$ is true for some $k \in \mathbb{N} \Rightarrow P(k+m)$ is true
then $P(n)$ is true $\forall n \in \mathbb{N}$.

(6) Odd-even M.I.

If
   (1) $P(1) \land P(2)$ is true;
   (2) $P(k)$ is true for some $k \in \mathbb{N} \Rightarrow P(k+2)$ is true
then $P(n)$ is true $\forall n \in \mathbb{N}$.
More difficult types of Mathematical Induction

(7) Backward M.I.

If 

1. $P(n)$ is true $\forall n \in A$, where $A$ is an infinite subset of $\mathbb{N}$;
2. $P(k)$ is true for some $k \in \mathbb{N} \Rightarrow P(k-1)$ is true

then $P(n)$ is true $\forall n \in \mathbb{N}$.

(8) Backward M.I. (variation) (more easily applied than (7))

If 

1. $P(1)$ is true;
2. $P(2^k)$ is true for some $k \in \mathbb{N} \Rightarrow P(2^{k+1})$ is true;
3. $P(k)$ is true for some $k \in \mathbb{N} \Rightarrow P(k-1)$ is true

then $P(n)$ is true $\forall n \in \mathbb{N}$.

(9) Different starting point

If 

1. $P(a)$ is true, where $a \in \mathbb{N}$;
2. $P(k)$ is true for some $k \in \mathbb{N}$, where $k \geq a \Rightarrow P(k+1)$ is true

then $P(n)$ is true $\forall n \in \mathbb{N}\{1, 2, \ldots, a-1\}$.

(10) Spiral M.I.

If 

1. $P(1)$ is true;
2. $P(k)$ is true for some $k \in \mathbb{N} \Rightarrow Q(k)$ is true
   $Q(k)$ is true for some $k \in \mathbb{N} \Rightarrow P(k+1)$ is true

then $P(n), Q(n)$ are true $\forall n \in \mathbb{N}$.

(11) Double M.I.

Double M.I. involves a proposition $P(m, n)$ with two variables $m, n$.

If 

1. $P(m, 1)$ and $P(1, n)$ is true $\forall m, n \in \mathbb{N}$;
2. $P(m+1, n)$ and $P(m, n+1)$ are true for some $m, n \in \mathbb{N} \Rightarrow P(m+1, n+1)$ is true

then $P(m, n)$ is true $\forall m, n \in \mathbb{N}$.
A Prime Number Theorem

Prove that the nth prime number \( p_n < 2^{2^n} \).

**Solution**

Let \( P(n) \) be the proposition : \( p_n < 2^{2^n} \).

For \( P(1) \), \( p_1 = 2 < 2^{2^1} \) \( \therefore \) \( P(1) \) is true.

Assume \( P(i) \) is true \( \forall \ i \ s.t. \ 1 \leq i \leq k \), i.e. \( p_1 < 2^{2^1} \), \( p_2 < 2^{2^2} \), ..., \( p_k < 2^{2^k} \) .... (*)

For \( P(k+1) \), Multiply all inequalities in (*), \( p_1p_2...p_k < 2^{2^2}2^{2^2}...2^{2^k} \)

\( p_1p_2...p_k + 1 \leq 2^{2^2}2^{2^2}...2^{2^k} = 2^{2^2+2^2+...+2^k} = 2^{2^{k+1}-2} < 2^{2^{k+1}} \)

\( \therefore \) For any prime factor \( p \) of \( p_1p_2...p_k + 1 \), we have \( p < 2^{2^{k+1}} \).

Since \( p_1, p_2, ..., p_k \) are not prime factor of \( p_1p_2...p_k + 1 \), we have \( p_k < p \) and hence \( p_{k+1} \leq p \).

\( \therefore \ p_{k+1} \leq p < 2^{2^{k+1}} \) \( \therefore \) \( P(k + 1) \) is true.

By the Second Principle of Mathematical Induction, \( P(n) \) is true \( \forall \ n \in \mathbb{N} \).

**Recursive formula**

Let \( \{a_n\} \) be a sequence of real numbers satisfying \( a_1 = 2 \), \( a_2 = 3 \) and \( a_{n+2} = 3a_{n+1} - 2a_n \).

Prove that \( a_n = 2^n - 1 \).

**Solution**

Let \( P(n) \) be the proposition : \( a_n = 2^n - 1 \).

For \( P(1) \land P(2) \), \( a_1 = 2 = 2^1 - 1 \), \( a_2 = 2 = 2^2 - 1 \) \( \therefore \) \( P(1) \land P(2) \) is true.

Assume \( P(k) \land P(k+1) \) is true for some \( k \in \mathbb{N} \).

\( i.e. \ a_k = 2^{k-1} + 1 \) .... (1)

\( a_{k+1} = 2^k + 1 \) .... (2)

For \( P(k+2) \), \( a_{k+2} = 3a_{k+1} - 2a_k = 3(2^{k+1} - 1) - 2(2^k + 1) = 2^{k+1} - 1 \)

\( \therefore \ P(k + 2) \) is true.

By the Second Principle of Mathematical Induction, \( P(n) \) is true \( \forall \ n \in \mathbb{N} \).

**Odd Even Mathematical Induction**

Let \( a_1 = 2 \), \( a_2 = 2 \) \( a_{n+2} = a_n + 1 \)

Prove that \( a_n = \frac{1}{2}(n+1) + \frac{1}{4}\left[1 + (-1)^n\right] \).

**Solution**
Let $P(n)$ be the proposition: 
$$a_n = \frac{1}{2}(n+1)+\frac{1}{4}[1+(-1)^n].$$

For $P(1)$, 
$$a_1 = \frac{1}{2}(1+1)+\frac{1}{4}[1+(-1)^1]$$
For $P(2)$, 
$$a_2 = \frac{1}{2}(2+1)+\frac{1}{4}[1+(-1)^2] \quad \therefore P(1) \land P(2) \text{ is true.}$$

Assume $P(k)$ is true for some $k \in \mathbb{N}$. i.e. 
$$a_k = \frac{1}{2}(k+1)+\frac{1}{4}[1+(-1)^k] \quad \ldots \quad (*)$$

For $P(k+2)$, 
$$a_{k+2} = a_k + 1 = \frac{1}{2}(k+1)+\frac{1}{4}[1+(-1)^k] + 1, \text{ by } (*)$$

$$= \frac{1}{2}((k+1) + 1) + \frac{1}{4}[1+(-1)^{k+1}]$$

$\therefore P(k+2)$ is true.

∴ By the Principle of Mathematical Induction, $P(n)$ is true $\forall n \in \mathbb{N}$.

**Backward Mathematical Induction**

Let $f(x)$ be a convex function defined on $[a, b]$, i.e. 
$$f(x) + f(x_2) \leq 2f\left(\frac{x_1 + x_2}{2}\right) \quad \text{for all } x_1, x_2 \in [a, b].$$

For each positive integer $n$, consider the statement: 
$$I(n) : \text{If } x_i \in [a, b], \quad i = 1, 2, \ldots, n, \text{ then } f(x_1) + \ldots + f(x_n) \leq nf\left(\frac{x_1 + \ldots + x_n}{n}\right).$$

(a) Prove by induction that $I(2^k)$ is true for every positive integer $k$.

(b) Prove that if $I(n)$ ($n \geq 2$) is true, then $I(n-1)$ is true.

(c) Prove that $I(n)$ is true for every positive integer $n$.

**Solution**

(a) $I(n) : \text{If } x_i \in [a, b], \quad i = 1, 2, \ldots, n, \text{ then } f(x_1) + \ldots + f(x_n) \leq nf\left(\frac{x_1 + \ldots + x_n}{n}\right)$

For $I(2^1)$, since it is given that 
$$f(x_1) + f(x_2) \leq 2f\left(\frac{x_1 + x_2}{2}\right). \quad \therefore I(2^1) \text{ is true.}$$

Assume $I(2^k)$ is true, i.e. 
$$f(x_1) + \ldots + f(x_{2^k}) \leq 2^k f\left(\frac{x_1 + \ldots + x_{2^k}}{2^k}\right) \quad \ldots \quad (1)$$

For $I(2^{k+1})$, 
$$f(x_1) + \ldots + f(x_{2^{k+1}}) + f(x_{2^k+1}) + \ldots + f(x_{2^{k+1}})$$

$$\leq 2^k f\left(\frac{x_1 + \ldots + x_{2^k}}{2^k}\right) + 2^k f\left(\frac{x_{2^k+1} + \ldots + x_{2^{k+1}}}{2^k}\right) = 2^k \left[ f\left(\frac{x_1 + \ldots + x_{2^k}}{2^k}\right) + f\left(\frac{x_{2^k+1} + \ldots + x_{2^{k+1}}}{2^k}\right) \right], \text{ by } (1)$$

$$= 2^k \left[ f\left(\frac{x_1 + \ldots + x_{2^k}}{2^k}\right) + f\left(\frac{x_{2^k+1} + \ldots + x_{2^{k+1}}}{2^k}\right) \right] \leq 2^{k+1} \left[ f\left(\frac{1}{2} \left( x_1 + \ldots + x_{2^k} + x_{2^k+1} + \ldots + x_{2^{k+1}} \right) \right) \right], \text{ by } I(2)$$

$$= 2^{k+1} \left[ f\left(\frac{x_1 + \ldots + x_{2^k} + x_{2^k+1} + \ldots + x_{2^{k+1}}}{2^{k+1}}\right) \right] \quad \therefore I(2^{k+1}) \text{ is true}$$
(b) Assume \( I(n) \) is true \((n \geq 2)\), i.e.

\[
\frac{x_1 + \ldots + x_n}{n} = \frac{n-1}{n} \left( \frac{x_1 + \ldots + x_{n-1}}{n-1} + \frac{x_n}{n-1} \right)
\]

Put \( x_n = \frac{x_1 + \ldots + x_{n-1}}{n-1} \), then

\[
f(x_1) + \ldots + f(x_{n-1}) + \frac{f(x_1 + \ldots + x_{n-1})}{n-1} \leq \frac{f(x_1 + \ldots + x_{n-1})}{n-1}
\]

\[
f(x_1) + \ldots + f(x_{n-1}) \leq (n-1)f\left(\frac{x_1 + \ldots + x_{n-1}}{n-1}\right)
\]

\[\therefore I(n-1) \text{ is also true.}\]

(c) \( \forall n \in \mathbb{N}, \exists (k \in \mathbb{N} \text{ and } r \in \mathbb{N}) \) such that \( n = 2^k - r \).

**Spiral Mathematical Induction**

Given a sequence \( \{a_n\} \) satisfying \( a_{2m-1} = 3m(m-1) + 1 \) and \( a_{2m} = 3m^2 \), where \( m \in \mathbb{N} \).

Let \( S_n = \sum_{i=1}^{n} a_i \), prove that

\[
\begin{align*}
S_{2m-1} &= \frac{1}{2} m(4m^2 - 3m + 1) \\
S_{2m} &= \frac{1}{2} m(4m^2 + 3m + 1)
\end{align*}
\]

**(1)**

**Solution**

Let \( P(m) \) be the proposition: \( S_{2m-1} = \frac{1}{2} m(4m^2 - 3m + 1) \)

\( Q(m) \) be the proposition: \( S_{2m} = \frac{1}{2} m(4m^2 + 3m + 1) \)

For \( P(1) \), \( S_1 = a_1 = 1 \). \( \therefore (1) \) is true for \( m = 1 \).

Assume \( P(k) \) is true for some \( k \in \mathbb{N} \), i.e. \( S_{2k-1} = \frac{1}{2} k(4k^2 - 3k + 1) \) \( \cdots \) \((*)\)

(a) For \( Q(k) \), \( S_{2k} = S_{2k-1} + a_{2k} = \frac{1}{2} k(4k^2 - 3k + 1) + 3k^2 = \frac{1}{2} k(4k^2 + 3k + 1) \). \( \therefore Q(k) \) is true.

(b) For \( P(k + 1) \),

\[
S_{2k+1} = S_{2k} + a_{2k+1} = \frac{1}{2} k(4k^2 + 3k + 1) + \frac{1}{2} [(4(k+1)^2 + 3(k+1) + 1] \\
= \frac{1}{2} \left(4(k+1)^2 + 9k^2 + 12k + 3\right) - (3k^2 + 6k + 3) + (k+1) \\
= \frac{1}{2} \left[4(k+1)^2 - 3(k+1) + (k+1)\right] = \frac{1}{2}(k+1)\left[4(k+1)^2 - 3(k+1) + 1\right]. \quad \therefore P(k+1) \text{ is true.}
\]

Since \( (1) \) \( P(1) \) is true.

\( (2) \) \( P(k) \) is true \( \Rightarrow Q(k) \) is true \( \Rightarrow P(k+1) \) is true

\( \therefore \) By the Principle of Mathematical Induction, \( P(n) \) is true \( \forall n \in \mathbb{N} \).

Since \( (1) \) \( P(1) \) is true. \( \Rightarrow \) \( Q(1) \) is true

\( (2) \) \( Q(k) \) is true \( \Rightarrow \) \( P(k+1) \) is true \( \Rightarrow Q(k+1) \) is true

\( \therefore \) By the Principle of Mathematical Induction, \( Q(n) \) is true \( \forall n \in \mathbb{N} \).
Mathematical Induction with parameter

Let \( f(a, 1) = \begin{cases} 1 & \text{when } a = 1 \\ 0 & \text{when } a > 1, \ a \in \mathbb{N} \end{cases} \)
and \( f(a, n+1) = \begin{cases} f(a, n) + 1 & \text{when } a = 1 \\ f(a, n) + f(a-1, n) & \text{when } a > 1, \ a \in \mathbb{N} \end{cases} \)

Prove that \( f(a, n) = \frac{n(n-1)...(n-a+1)}{a!} \)

Solution

Let \( P(n) \) be the proposition: \( f(a, n) = \frac{n(n-1)...(n-a+1)}{a!} \) \( \ldots \) (1)

(1) For \( P(1) \), there are two cases:

When \( a = 1 \), \( \text{L.H.S.} = f(1, 1) = 1 \) \( \Rightarrow \text{R.H.S.} = \frac{1}{1!} = 1 \)
When \( a > 1 \), \( \text{L.H.S.} = f(a, 1) = 0 \) \( \Rightarrow \text{R.H.S.} = \frac{1(1-1)...(1-a+1)}{a!} = 0 \). \( \therefore \) \( P(1) \) is true.

(2) Assume \( P(k) \) is true for some \( k \in \mathbb{N} \), i.e. \( f(a, k) = \frac{k(k-1)...(k-a+1)}{a!} \) \( \ldots \) (2)

For \( P(k + 1) \), there are also two cases:

When \( a = 1 \), \( \text{L.H.S.} = f(a, k + 1) = f(a, k) + 1 = \frac{k}{1!} + 1 = k + 1 = \frac{k + 1}{1!} = \text{R.H.S.} \)
When \( a > 1 \), \( \text{L.H.S.} = f(a, k) + f(a-1, k) \)
\( = \frac{k(k-1)...(k-a+1)}{a!} + \frac{k(k-1)...(k-a+2)}{(a-1)!} \) \( \), by (2), \( f(a,k) \) and \( f(a-1, k) \) hold.
\( = \frac{k(k-1)...(k-a+2)}{a!}[(k-a+1)+a] \)
\( = \frac{(k+1)k(k-1)...(k-a+2)}{a!} = \text{R.H.S.} \)

\( \therefore \) \( P(k + 1) \) is true.

\( \therefore \) By the Principle of Mathematical Induction, \( P(n) \) is true \( \forall \ n \in \mathbb{N} \).

Comment If the proposition with natural number \( n \) contains a parameter \( a \), then we need to apply mathematical induction for all values of \( a \).
Double Mathematical Induction

Prove that the number of non-negative integral solution sets of the equation
\[ x_1 + x_2 + \ldots + x_m = n \quad , \quad m , n \in \mathbb{N} . \]
is
\[ f(m, n) = \frac{(n + m - 1)!}{n!(m - 1)!} \quad .... (1) \]

Solution

Let \( P(m, n) \) be the given proposition.

(a) For \( P(1, n) \), The only non-negative integral solution set of the equation \( x_1 = n \) is only itself.
In \( (1) \), 
\[ f(1, n) = \frac{(n + 1 - 1)!}{n!(1 - 1)!} = 1 . \]
\( \therefore P(1, n) \) is true.

For \( P(m, 1) \), The non-negative integral solution sets of the equation
\[ x_1 + x_2 + \ldots + x_m = 1 \]
are \( (1, 0, 0 \ldots, 0) , (0, 1, 0, \ldots) , \ldots, (0, 0, 0, \ldots, 1) . \)
There are \( m \) sets of solution altogether.
In \( (1) \), 
\[ f(m, 1) = \frac{(1 + m - 1)!}{1!(m - 1)!} = m . \]
\( \therefore P(m, 1) \) is true.

(b) Assume \( P(m, n+1) \) and \( P(m + 1, n) \) are true for some \( m , n \in \mathbb{N} \ldots \) i.e
the number of non-negative integral solution sets of the equations:
\[ x_1 + x_2 + \ldots + x_m = n + 1 \quad .... (2) \]
\[ x_1 + x_2 + \ldots + x_m + x_{m+1} = n \quad .... (3) \]
are
\[ f(m, n+1) = \frac{(n + m)!}{(n + 1)!(m - 1)!} \quad \text{ and } \quad f(m+1, n) = \frac{(n + m)!}{n!m!} \quad \text{ respectively} . \]

For \( P(m+1, n+1) \), The non-negative integral solution sets of the equation:
\[ x_1 + x_2 + \ldots + x_m + x_{m+1} = n + 1 \quad .... (4) \]
may be divided into two parts : \( x_{m+1} = 0 \) or \( x_{m+1} > 0 \).
(i) For \( x_{m+1} = 0 \), equation \( (4) \) becomes equation \( (2) \), and the number of non-negative integral solution
sets is
\[ f(m, n+1) = \frac{(n + m)!}{(n + 1)!(m - 1)!} . \]
(ii) For \( x_{m+1} > 0 \), replace \( x_{m+1} \) by \( x_{m+1} + 1 \) and equation \( (4) \) becomes:
\[ x_1 + x_2 + \ldots + x_m + x_{m+1} = n , \] and the number of non-negative integral solution
sets is
\[ f(m+1, n) = \frac{(n + m)!}{n!m!} . \]
\( \therefore \) The total number of non-negative integral solution sets is
\[ \frac{(n + m)!}{(n + 1)!(m - 1)!} + \frac{(n + m)!}{n!m!} = \frac{(n + m)!}{(n + 1)!m!} - \frac{[(n + 1) + (m + 1) - 1]!}{(n + 1)(m + 1) - 1)!} . \]
\( \therefore P(m+1, n+1) \) is also true.
\( \therefore \) By the Principle of Mathematical Induction, \( P(m, n) \) is true \( \forall m, n \in \mathbb{N} \).