QUEEN'S COLLEGE  
Yearly Examination 2005-2006  
Pure Mathematics  

Secondary 6 E , S  

Date: 21st June 2006  
Time: 8 : 30 – 11 : 30  

Instructions:  
(1) Answer ALL questions in Section A and section B.  
(2) All workings must be clearly shown.  
(3) Unless otherwise specified, numerical answers must be exact.  
(4) The diagrams in this paper are not necessarily drawn to scale.  

FORMULA FOR REFERENCE  

\[
\begin{align*}
\sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\
\cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \\
\tan(A \pm B) &= \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \\
\sin A + \sin B &= 2 \sin \frac{A + B}{2} \cos \frac{A - B}{2} \\
\sin A - \sin B &= 2 \cos \frac{A + B}{2} \sin \frac{A - B}{2} \\
\cos A + \cos B &= 2 \cos \frac{A + B}{2} \cos \frac{A - B}{2} \\
\cos A - \cos B &= -2 \sin \frac{A + B}{2} \sin \frac{A - B}{2} \\
2 \sin A \cos B &= \sin(A + B) + \sin(A - B) \\
2 \cos A \cos B &= \cos(A + B) + \cos(A - B) \\
2 \sin A \sin B &= \cos(A - B) - \cos(A + B)
\end{align*}
\]
Section A (80 marks):

Answer ALL questions in this section.

1. Evaluate
   \[ \lim_{x \to 1} \left( \pi - x \right) \tan \frac{x}{2}, \]
   \[ \lim_{x \to 1} \left( \frac{x}{\sin x} \right)^3. \]

   \[ \text{(8 marks)} \]

2. Consider the following system of linear equations:

   \[
   \begin{align*}
   3x - y + z &= 1 \\
   2x - 4y - 5z &= 1 \\
   4x + 2y + 7z &= c
   \end{align*}
   \]

   where \( c \in \mathbb{R} \).

   Suppose \((*)\) is consistent. Find \( c \) and solve \((*)\). \( \text{(8 marks)} \)

3. By using Mean Value Theorem, show that for \( x > 0, \ 0 < a \leq 1 \)

   \[ 1 + x < e^x < 1 + ex. \]

   \[ \text{(12 marks)} \]

4. Let \( f(x) = \begin{cases} x^2 + ax + b & \text{when } x \leq 1, \\ \sin \frac{nx}{n} & \text{when } x > 1. \end{cases} \)

   If \( f \) is differentiable at \( 1 \), find \( a \) and \( b \). \( \text{(12 marks)} \)

5. Let \( y = a \cos (\ln x) + b \sin (\ln x) \) where \( x > 0 \)

   (a) Show that

   \[ x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = 0. \]

   (b) Hence deduce that

   \[ x^2 \frac{d^{n+1} y}{dx^{n+1}} + (2n + 1)x \frac{d^{n+1} y}{dx^{n+1}} + (n^2 + 1) \frac{d^n y}{dx^n} = 0 \]

   \[ \text{(12 marks)} \]
6. Let \( A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \), \( B = \begin{pmatrix} \sin \theta & -\cos \theta \\ \cos \theta & \sin \theta \end{pmatrix} \) where \( 0 \leq \theta \leq \frac{\pi}{2} \).

(a) Show that \( BAB^{-1} = \begin{pmatrix} 2 - \sin 2\theta & -\cos 2\theta \\ -\cos 2\theta & 2 + \sin 2\theta \end{pmatrix} \).

(b) It is given that \( BAB^{-1} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \).

Find the values of \( \theta \), \( a \) and \( b \).

(c) Use the above results to evaluate \( A^n \) in terms of \( n \). (14 marks)

7. The sequence of positive real numbers \( x_n \) is defined by

\[
x_1 = 1,
\]

\[
x_{n+1} = \frac{1 + 2x_n}{1 + x_n}, \quad n \geq 1.
\]

(a) Show that \( x_{n+1} < 2 \) for all \( n \geq 1 \).

(b) By mathematical induction, show that \( \{x_n\} \) is monotonically increasing for all \( n \geq 1 \).

(c) Show that it is convergent and find its limit. (14 marks)
Section B (120 marks):

Answer ALL questions in this section.

8. The polynomials \( P_0(x), P_1(x), P_2(x), \ldots \) are defined by

\[
P_0(x) = 1, \quad P_1(x) = x, \quad P_r(x) = \frac{x(x-1) \cdots (x-r+1)}{r!}, \quad \text{where} \quad r = 2, 3, \ldots
\]

(a) Show that \( P_r(k) \) is an integer for \( r \geq 0 \) and for any integer \( k \).

[Hint: For \( r \geq 2 \), consider the cases \( r \leq k \), \( 0 \leq k < r \) and \( k < 0 \) and use the fact that the binomial coefficients \( \binom{r}{k} \) are integers.]

(12 marks)

(b) Let \( P(x) = \sum_{r=0}^{n} a_r P_r(x) \), where \( a_0, a_1, \ldots, a_n \) are constants.

If \( a_0, a_1, \ldots, a_{n-1} \) are integers but \( a_n(0 < m \leq n) \) is not, show that \( P(m) \) is not an integer.

Deduce that if \( P(0), P(1), \ldots, P(n) \) are integers, then

(i) \( a_0, a_1, \ldots, a_n \) are integers,

(ii) \( P(k) \) is an integer for any integer \( k \).

(14 marks)

(c) Find a polynomial \( Q(x) = ax^2 + bx + c \) such that \( Q(k) \) is an integer for any integer \( k \), but not all of \( a, b \) and \( c \) are integers.

(4 marks)
9. Let \( f(x) = e^x \) \((x \neq 0)\).

(a) (i) Find \( \lim_{x \to 0^+} f(x) \) and \( \lim_{x \to 0^-} f(x) \).

(ii) Find \( f'(x) \) and \( f''(x) \) for \( x \neq 0 \). (8 marks)

(b) Determine the range of values of \( x \) for each of the following cases:

(i) \( f'(x) > 0 \),
(ii) \( f'(x) < 0 \),
(iii) \( f''(x) > 0 \),
(iv) \( f''(x) < 0 \). (8 marks)

(c) Find the relative extreme point(s) and point(s) of inflexion of \( f(x) \). (4 marks)

(d) Find the asymptote(s) of the graph of \( f(x) \). (6 marks)

(e) Sketch of the graph \( f(x) \). (4 marks)

10. Let \( \{a_n\} \) be a sequence of real numbers such that \( 0 < a_n < 1 \) and \( a_{n+1} = \sin(a_n) \) for all \( n \in \mathbb{N} \).

(a) Making use of the fact that \( \sin x < x \) for \( 0 < x < 1 \), show that \( \{a_n\} \) is a monotonic bounded sequence. Hence find its value. (14 marks)

(b) (i) Show that \( \lim_{n \to \infty} \frac{x^2 - \sin^2 x}{x^2 \sin^2 x} = \frac{1}{3} \).

(ii) Hence find \( \lim_{n \to \infty} \left( \frac{1}{a_{n+1}^2} - \frac{1}{a_n^2} \right) \). (10 marks)

(c) It is known that if \( \lim_{x \to \infty} x_n \) exists and equals \( L \), then \( \lim_{n \to \infty} \frac{\sum_{i=1}^{n} x_i}{n} \) also exists and equals \( L \).

Use this fact, or otherwise, to show that \( \lim_{n \to \infty} \left( n a_n^2 \right) \) exists and find its value. (6 marks)
11. Let $f: \mathbb{R} \to \mathbb{R}$ be a function such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

(a) Show that

(i) $f(0) = 0$,

(ii) $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

(iii) $f(nx) = nf(x)$ for all $n \in \mathbb{Z}$ and $x \in \mathbb{R}$.

(b) Show that if there exists $K > 0$ such that $f(x) < K$ for all $x \in \mathbb{R}$ then $f(x) = 0$ for all $x \in \mathbb{R}$.

(c) Suppose there exists $K > 0$ such that $f(x) < K$ for all $x \in [0, 1)$.

Let $g(x) = f(x) - f(1)x$ for all $x \in \mathbb{R}$.

Show that, for all $x, y \in \mathbb{R}$,

(i) $g(x+y) = g(x) + g(y)$,

(ii) $g(x+1) = g(x)$,

(iii) $g(x) < K + |f(1)|$.

Hence, or otherwise, show that $f(x) = f(1)x$ for all $x \in \mathbb{R}$.

END OF PAPER