Queen's College
Mock Examination 2006 – 2007

ADDITIONAL MATHEMATICS
Question Answer Book

Date: 13th March 2007
Time: 11:00 am – 1:30 pm (2.5 hours)
This paper must be answered in English

1. Write your class and class number in the space provided on page 1.
2. This paper consists of TWO sections, Section A and Section B. Section A carries 62 marks and Section B carries 48 marks.
3. Answer ALL questions in Section A. Write your answers in the spaces provided in this Question-Answer Book. Do not write in the margins. Graph paper and supplementary answer sheets will be supplied on request. Write your class and class number on each sheet, and fasten them with a string INSIDE this book.
4. Answer any FOUR questions in Section B. Write your answers in the answer book provided.
5. The Question-Answer Book and the answer book must be handed in separately at the end of the examination.
6. All working must be clearly shown.
7. Unless otherwise specified, numerical answers must be exact.
8. In this paper, vectors may be represented by bold-type letters such as \( \mathbf{u} \), but candidates are expected to use appropriate symbols such as \( \vec{u} \) in their working.
9. The diagrams in the paper are not necessarily drawn to scale.

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<th>Section A Question No.</th>
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Section A Total

Mock Examination AM 06 - 07
FORMULA FOR REFERENCE

\[
\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B
\]
\[
\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B
\]
\[
\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}
\]
\[
2 \sin A \cos B = \sin(A + B) + \sin(A - B)
\]
\[
2 \cos A \cos B = \cos(A + B) + \cos(A - B)
\]
\[
2 \sin A \sin B = \cos(A - B) - \cos(A + B)
\]

\[
\sin A + \sin B = 2 \sin \frac{A + B}{2} \cos \frac{A - B}{2}
\]
\[
\sin A - \sin B = 2 \cos \frac{A + B}{2} \sin \frac{A - B}{2}
\]
\[
\cos A + \cos B = 2 \cos \frac{A + B}{2} \cos \frac{A - B}{2}
\]
\[
\cos A - \cos B = -2 \sin \frac{A + B}{2} \sin \frac{A - B}{2}
\]

SECTION A (62 marks)
Answer ALL questions in this section and write your answers in the space provided in this Question-Answer Book.

1. Let \( f(x) = x \sin x \). Evaluate \( f\left(\frac{\pi}{2}\right) \) \( \text{ (3 marks) } \)

2. In the binomial expansion of \( \left(1 + \frac{x}{n}\right)^n \) in ascending powers of \( x \), the coefficient of \( x^2 \) is \( \frac{7}{16} \). Given that \( n \) is a positive integer,

   (a) find the value of \( n \),
   
   (b) evaluate the coefficient of \( x^3 \) in the expansion. \( \text{ (5 marks) } \)
3. Solve \( \left| \frac{2x-1}{5} \right| = 9 \). (3 marks)

4. \( \alpha \) and \( \beta \) are the roots of \( x^2 + px - 5 = 0 \), while \( \alpha^2 \) and \( \beta^2 \) are the roots of \( x^2 - 19x + q = 0 \). Find the possible values of \( p \) and \( q \). (5 marks)

5. Prove by mathematical induction that if \( n \) is a positive integer, \( 3^{2n} - 1 \) is divisible by 8. (5 marks)
6. The line $y = x$ and the circle $x^2 + y^2 - 2y = 0$ intersect at the points $A$ and $B$. Write down the equation of the family of circles passing through $A$ and $B$.

Hence find the equations of the two circles passing through these two points and with radius $\sqrt{5}$. (6 marks)

7. The slope at any point $(x, y)$ of a curve is given by

$$\frac{dy}{dx} = \sqrt{x}(4 - x)$$

If the curve passes through the point $(1, 1)$, find the equation of the curve. (5 marks)
8. Find the general solution of the equation
\[ \sin \left( x - \frac{\pi}{4} \right) + \sqrt{3} \sin \left( x + \frac{\pi}{4} \right) = 2. \]

(6 marks)

9. Given the curve \( C : x^2 + 4xy + 5y^2 = 1 \), find \( \frac{dy}{dx} \).

Hence find the equations of the two tangents to \( C \) which are parallel to the line \( y = -\frac{1}{2}x \).

(6 marks)
10. Let \( f(x) = x^2 + 4mx + 4m + 15 \), where \( m \) is a constant.

Find the discriminant of the equation \( f(x) = 0 \).

Hence, or otherwise, find the range of values of \( m \) so that \( f(x) > 0 \) for all real values of \( x \). (4 marks)

11. Let \( \overrightarrow{OA} = i + 2j, \overrightarrow{OB} = 2i - 4j \) and \( P \) be a point on \( AB \) such that \( AP : PB = k : 1 \), where \( k > 0 \) is a constant.

(a) Find \( \overrightarrow{OP} \) in terms of \( k, i \) and \( j \).

(b) Find \( \overrightarrow{OA} \cdot \overrightarrow{OP} \) and \( \overrightarrow{OB} \cdot \overrightarrow{OP} \) in terms of \( k \).

(c) If \( \overrightarrow{OP} \) bisects \( \angle AOB \), find the value of \( k \). (7 marks)
Figure 1 shows the graph of the functions \( y = k \left( x - \frac{\pi}{2} \right)^2 \) and \( y = a \sin x + b \), where \( a, b \) and \( k \) are constants.

(a) Find the values of \( a, b \) and \( k \).

(b) Find the area of the shaded region. (7 marks)
SECTION B (48 marks)

Answer any **FOUR** questions in this section. Each question carries 12 marks.

Write your answers in the answer book.

13.

![Figure 2](image)

In Figure 2, a straight line $L : y = mx + 2$, where $m > 0$, is given. $P$ is the foot of the perpendicular from the origin to $L$. $C$ is the in-centre of the triangle bounded by $L$, the $x$-axis and the $y$-axis.

(a) Find, in terms of $m$, the coordinates of $P$. 

(b) (i) Find the equation of the locus of $P$ as $m$ varies. Sketch the locus with brief explanations.

(ii) By considering the coordinates of $C$, find the equation of the locus of $C$ as $m$ varies. Sketch the locus with brief explanations.
(a) In Figure 3, \( OAB \) is a triangle, and \( C, D \) are points on \( OA \) and \( OB \) respectively. \( OE \) is produced to meet \( AB \) at \( F \). Let \( \overline{OA} = \mathbf{a}, \overline{OB} = \mathbf{b} \), \( OC = \lambda \overline{OA}, \ OD = \gamma \overline{OB}, \ AE : ED = k : 1 \) and \( BE : EC = h : 1 \).

(i) Find \( \overline{OE} \) in terms of

(1) \( h, \lambda, \mathbf{a} \) and \( \mathbf{b} \) only.

(2) \( k, \gamma, \mathbf{a} \) and \( \mathbf{b} \) only.

(ii) Find \( \lambda \) and \( \gamma \) in terms of \( h \) and \( k \).

(iii) Show that
\[
\frac{AF}{FB} = \frac{\gamma(1 - \lambda)}{\lambda(1 - \gamma)}.
\] (8 marks)

(b) In Figure 4, \( PQR \) is a triangle, and \( S \) and \( T \) are points on \( PQ \) and \( PR \) respectively. \( QT \) and \( RS \) meet at \( U \). Suppose \( PS : SQ = 3 : 2 \) and \( PT : TR = 4 : 3 \). Using (a)(i) and (ii), or otherwise, find \( QU : UT \) and \( RU : US \).

(4 marks)
In Figure 5, \(CDEF\) and \(ABFE\) are two rectangular planes perpendicular to each other. Let \(AB = 2a\), \(BC = a\), \(\angle CBF = 30^\circ\). \(X\) is the mid-point of \(AB\). \(P\) is a point on \(AC\) such that \(\frac{PC}{AC} = k\); \(Q\) is a point on \(XC\) such that \(\frac{QC}{XC} = h\). \(PP'\) and \(QQ'\) are the horizontal lines through \(P\) and \(Q\), meeting the line \(CF\) at \(P'\) and \(Q'\) respectively.

(a) Show that \(PC = \sqrt{5}ka\) and \(QC = \sqrt{2}ha\).  

(b) By considering \(\triangle ACX\), show that \(\cos \angle ACX = \frac{3}{\sqrt{10}}\).  

(c) Given that the area of \(\triangle PCQ = \frac{1}{8}a^2\).

(i) Find the value of \(hk\).

(ii) If, in addition, \(P'Q'\) is of length \(\frac{1}{8}a\), find \(|k - h|\).

Hence, or otherwise, find the value of \(k\) when \(h > k\).
In Figure 6, the shaded region is bounded by the curve \( y = x^2 - k \), the \( x \)-axis, the \( y \)-axis and the line \( y = -h \), where \( 0 < h \leq k \). The shaded region is revolved about the \( y \)-axis. Let \( V \) be the volume of the solid generated. Show that

\[
V = \frac{\pi}{2} (2kh - h^3) \text{ cubic units.}
\]

Figure 7 shows a vessel with a lid. The vessel is in the shape of the solid generated in (a) when \( h = k \). The radius of the lid is \( \sqrt{5} \) cm and the height of the vessel is 5 cm. Initially, the vessel contains oil to the depth 3 cm.

(i) Find the volume of the oil in the vessel.

(ii) The vessel is then inverted and placed on a horizontal plane (Figure 8) and the new height of the oil in the vessel is \( H \) cm. Find \( H \).

(iii) It is found that the oil is leaking out at a rate \( \pi \text{ cm}^3 \) per second after inverting the vessel. By b(ii), find the initial rate of change of \( H \).

\[\text{(9 marks)}\]
Figure 9 shows that a tent which consists of two parts, a hemisphere and a cylinder (curved surface only). The base radius and the height of the cylinder are $r$ metres and $h$ metres respectively. The radius of the hemisphere is $r$ metres. The volume of the tent is $9\pi$ cubic metres.

(a) Find $h$ in terms of $r$. \hspace{1cm} (2 marks)

(b) The cost per square metre in producing the hemispherical part and cylindrical part are $\$ k$ and $\$ 1$ respectively. It is given that $0 < r \leq 2$. Let $\$ C$ be the cost in producing the tent.

(i) Show that $C = \frac{18\pi}{r} + \frac{2(3k-2)\pi}{3} r^2$.

(ii) Find $\frac{dC}{dr}$ in terms of $r$.

(iii) Find the value of $r$ when $C$ is minimum in each of the following cases:

1. $k = 2$, \hspace{1cm} (10 marks)
2. $k = \frac{1}{2}$.