$\int \frac{dx}{x^3+1}$, $\int \frac{dx}{x^4+1}$, $\int \sqrt{\tan\theta}\, d\theta$, $\int \sqrt{\cot\theta}\, d\theta$

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1. $\int \frac{dx}{x^3+1} = \int \frac{dx}{(x+1)(x^2-x+1)}$

$= \frac{1}{3}\int \left(\frac{1}{x+1} - \frac{x-2}{x^2-x+1}\right) dx$ (Break integrand by Partial Fraction)

$$= \frac{1}{3}\int \frac{dx}{x+1} - \frac{1}{6}\int \frac{(2x-1)-3}{x^2-x+1} dx$$

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$$= \frac{1}{3}\int \frac{dx}{x+1} - \frac{1}{6}\int \frac{2x-1}{x^2-x+1} dx + \frac{1}{2}\int \frac{1}{x^2-x+1} dx$$

$$= \frac{1}{3}\int \frac{d(x+1)}{x+1} - \frac{1}{6}\int \frac{d(x^2-x+1)}{x^2-x+1} + \frac{1}{2}\int \frac{1}{\left(x-\frac{1}{2}\right)^2 + \left(\sqrt{\frac{3}{4}}\right)^2} dx$$

$$= \frac{1}{3}\ln(x+1) - \frac{1}{6}\ln(x^2-x+1) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

For $I = \int \frac{1}{\left(x-\frac{1}{2}\right)^2 + \left(\sqrt{\frac{3}{4}}\right)^2} dx$, let $x - \frac{1}{2} = \sqrt{\frac{3}{4}}\tan\theta$, $dx = \sqrt{\frac{3}{4}}\sec^2\theta\, d\theta$

$$\therefore \quad I = \int \frac{\sqrt{\frac{3}{4}}\sec^2\theta\, d\theta}{\frac{3}{4}\sec^2\theta} = \frac{2}{\sqrt{3}}\int d\theta$$

$$= \frac{2}{\sqrt{3}}\theta + C = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

2. Method 1

$$I = \int \frac{dx}{x^4+1} = \int \frac{dx}{(x^2+1)^2 - 2x^2} = \int \frac{dx}{(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)}$$

The partial fraction is clumsy:

$$\frac{1}{(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)} = \frac{Ax+B}{x^2+\sqrt{2}x+1} + \frac{Cx+D}{x^2-\sqrt{2}x+1}$$

We get:

$$(Ax+B)\left(x^2-\sqrt{2}x+1\right) + (Cx+D)\left(x^2+\sqrt{2}x+1\right) = 1$$

We need to solve for A, B, C, D with 4 equations involving surds.

To avoid this, please study the following:

(a) $(x^2+\sqrt{2}x)(x^2-\sqrt{2}x)-(x^2-\sqrt{2}x)(x^2+\sqrt{2}x)=0$

$x(x+\sqrt{2})(x^2-\sqrt{2}x)-x(x-\sqrt{2})(x^2+\sqrt{2}x)=0$

Cancel x in (a), we get:

(b) $(x+\sqrt{2})(x^2-\sqrt{2}x)-(x-\sqrt{2})(x^2+\sqrt{2}x)=0$

(c) $(x+\sqrt{2})1-(x-\sqrt{2})1=2\sqrt{2}$

Adding (b) and (c), we get:

(d) $(x+\sqrt{2})(x^2-\sqrt{2}x+1)-(x-\sqrt{2})(x^2+\sqrt{2}x+1)=2\sqrt{2}$

(e) $\frac{1}{(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)}=\frac{1}{2\sqrt{2}}\left(\frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1}-\frac{x-\sqrt{2}}{x^2-\sqrt{2}x+1}\right)$

$$\therefore\quad I=\frac{1}{2\sqrt{2}}\int\frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1}dx-\frac{1}{2\sqrt{2}}\int\frac{x-\sqrt{2}}{x^2-\sqrt{2}x+1}dx$$

$$=\frac{1}{4\sqrt{2}}\int\frac{d(x^2+\sqrt{2}x+1)}{x^2+\sqrt{2}x+1}+\frac{1}{4}\int\frac{dx}{x^2+\sqrt{2}x+1}-\frac{1}{4\sqrt{2}}\int\frac{d(x^2-\sqrt{2}x+1)}{x^2-\sqrt{2}x+1}+\frac{1}{4}\int\frac{dx}{x^2-\sqrt{2}x+1}$$

$$=\frac{1}{4\sqrt{2}}\ln\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right)+\frac{1}{4}\int\frac{dx}{\left(x+\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2}+\frac{1}{4}\int\frac{dx}{\left(x-\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2}$$

$$=\frac{\sqrt{2}}{8}\ln\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right)+\frac{\sqrt{2}}{4}\tan^{-1}(\sqrt{2}x+1)+\frac{\sqrt{2}}{4}\tan^{-1}(\sqrt{2}x-1)+C$$

$$=\frac{\sqrt{2}}{8}\ln\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right)+\frac{\sqrt{2}}{4}\left[\tan^{-1}(\sqrt{2}x+1)+\tan^{-1}(\sqrt{2}x-1)\right]+C$$

$$=\frac{\sqrt{2}}{8}\ln\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right)+\frac{\sqrt{2}}{4}\left[\tan^{-1}\frac{(\sqrt{2}x+1)+(\sqrt{2}x-1)}{1-(\sqrt{2}x+1)(\sqrt{2}x-1)}\right]+C$$

$$=\frac{\sqrt{2}}{8}\ln\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right)+\frac{\sqrt{2}}{4}\tan^{-1}\frac{\sqrt{2}x}{1-x^2}+C \qquad \text{....} \quad \textbf{(1)}$$

3. Method 2

$$I=\int\frac{dx}{x^4+1}=\frac{1}{2}\int\frac{(x^2+1)-(x^2-1)dx}{x^4+1}$$

$$=\frac{1}{2}\int\frac{x^2+1}{x^4+1}dx-\frac{1}{2}\int\frac{x^2-1}{x^4+1}$$

$$=\frac{1}{2}I_1-\frac{1}{2}I_2$$

For $\quad I_1 = \int \frac{x^2+1}{x^4+1} dx = \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}} dx$

Let $\quad u = x - \frac{1}{x}, \quad du = \left(1 + \frac{1}{x^2}\right) dx$

And $\quad x^2 + \frac{1}{x^2} = \left(x - \frac{1}{x}\right)^2 + 2 = u^2 + 2$

$$\therefore \quad I_1 = \int \frac{du}{u^2+2} = \frac{\sqrt{2}}{2} \tan^{-1} \frac{u}{\sqrt{2}} + C_1$$

$$= \frac{\sqrt{2}}{2} \tan^{-1} \frac{1}{\sqrt{2}} \left(x - \frac{1}{x}\right) + C_1 \quad = \frac{\sqrt{2}}{2} \tan^{-1} \frac{x^2-1}{\sqrt{2}x} + C_1$$

For $\quad I_2 = \int \frac{x^2-1}{x^4+1} dx = \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}} dx$

Let $\quad u = x + \frac{1}{x}, \quad du = \left(1 - \frac{1}{x^2}\right) dx$

And $\quad x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2 = u^2 - 2$

$$\therefore \quad I_2 = \int \frac{du}{u^2-2} = \int \frac{du}{(u-\sqrt{2})(u+\sqrt{2})}$$

$$= \frac{1}{2\sqrt{2}} \left(\int \frac{du}{u-\sqrt{2}} - \int \frac{du}{u+\sqrt{2}} \right)$$

$$= \frac{\sqrt{2}}{4} \ln \left(\frac{u-\sqrt{2}}{u+\sqrt{2}} \right) + C_2$$

$$= \frac{\sqrt{2}}{4} \ln \left(\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}} \right) + C_2$$

$$= \frac{\sqrt{2}}{4} \ln \left(\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1} \right) + C_2$$

$$\therefore \quad I = \frac{1}{2} I_1 - \frac{1}{2} I_2 = \frac{\sqrt{2}}{4} \tan^{-1} \frac{x^2-1}{\sqrt{2}x} - \frac{\sqrt{2}}{8} \ln \left(\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1} \right) + C$$

$$= \frac{\sqrt{2}}{4} \tan^{-1} \frac{x^2-1}{\sqrt{2}x} + \frac{\sqrt{2}}{8} \ln \left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1} \right) + C \qquad \text{.... } \mathbf{(2)}$$

4. The results of Method 1 and Method 2 differ by a constant.

Just observe : $\quad \tan^{-1} \frac{x^2-1}{\sqrt{2}x} - \tan^{-1} \frac{\sqrt{2}x}{1-x^2} = \tan^{-1} \frac{\frac{x^2-1}{\sqrt{2}x} - \frac{\sqrt{2}x}{1-x^2}}{1+\left(\frac{x^2-1}{\sqrt{2}x}\right)\left(\frac{\sqrt{2}x}{1-x^2}\right)} = \tan^{-1}\infty = \frac{\pi}{2}$

5. $\int \sqrt{\tan\theta}\,d\theta,\ \int \sqrt{\cot\theta}\,d\theta$

$$\int \frac{dx}{x^4+1} = \frac{\sqrt{2}}{4}\tan^{-1}\frac{x^2-1}{\sqrt{2}x} + \frac{\sqrt{2}}{8}\ln\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right) + C$$

(a) Put $x = \sqrt{\tan\theta}$, $dx = \frac{\sec^2\theta}{2\sqrt{\tan\theta}}d\theta$

$$\int \frac{dx}{x^4+1} = \int \frac{\sec^2\theta}{(\tan^2\theta+1)2\sqrt{\tan\theta}}d\theta = \frac{1}{2}\int \sqrt{\cot\theta}\,d\theta$$

$$\therefore \int \sqrt{\cot\theta}\,d\theta = \frac{\sqrt{2}}{2}\tan^{-1}\frac{\tan\theta-1}{\sqrt{2\tan\theta}} + \frac{\sqrt{2}}{4}\ln\left(\frac{\tan\theta+\sqrt{2\tan\theta}+1}{\tan\theta-\sqrt{2\tan\theta}+1}\right) + C$$

(b) Put $x = \sqrt{\cot\theta}$, $dx = \frac{-\csc^2\theta}{2\sqrt{\cot\theta}}d\theta$

$$\int \frac{dx}{x^4+1} = \int \frac{-\csc^2\theta}{(\cot^2\theta+1)2\sqrt{\cot\theta}}d\theta = -\frac{1}{2}\int \sqrt{\tan\theta}\,d\theta$$

$$\int \sqrt{\tan\theta}\,d\theta = -\frac{\sqrt{2}}{2}\tan^{-1}\frac{\cot\theta-1}{\sqrt{2\cot\theta}} - \frac{\sqrt{2}}{4}\ln\left(\frac{\cot\theta+\sqrt{2\cot\theta}+1}{\cot\theta-\sqrt{2\cot\theta}+1}\right) + C$$