Rearrangement Inequality

Yue Kwok Choy

The rearrangement inequality (also known as permutation inequality) is easy to understand and yet a powerful tool to handle inequality problems.

Definition

Let \( a_1 \leq a_2 \leq \ldots \leq a_n \) and \( b_1 \leq b_2 \leq \ldots \leq b_n \) be any real numbers.

(a) \( S = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n \) is called the Sorted sum of the numbers.

(b) \( R = a_1 b_n + a_2 b_{n-1} + \ldots + a_n b_1 \) is called the Reversed sum of the numbers.

(c) Let \( c_1, c_2, \ldots, c_n \) be any permutation of the numbers \( b_1, b_2, \ldots, b_n \).

\[ P = a_1 c_1 + a_2 c_2 + \ldots + a_n c_n \]

is called the Permutated sum of the numbers.

Rearrangement inequality

\[ S \geq P \geq R \]

Proof

(a) Let \( P(n) \) be the proposition : \( S \geq P \).

\[ P(1) \text{ is obviously true.} \]

Assume \( P(k) \) is true for some \( k \in \mathbb{N} \).

For \( P(k + 1) \), Since the \( c \)'s are the permutations of the \( b \)'s, suppose \( b_{k+1} = c_i \) and \( c_{k+1} = b_j \)

\[ (a_{k+1} - a_i)(b_{k+1} - b_j) \geq 0 \]

\[ \Rightarrow a_i b_j + a_{k+1} b_{k+1} - a_i b_{k+1} - a_{k+1} b_j \]

\[ \Rightarrow a_i b_j + a_{k+1} b_{k+1} \geq a_i c_i + a_{k+1} c_{k+1} \]

So in \( P \), we may switch \( c_i \) and \( c_{k+1} \) to get a possibly larger sum.

After switching of these terms, we come up with the inductive hypothesis \( P(k) \).

\[ \therefore P(k + 1) \text{ is also true.} \]

By the principle of mathematical induction, \( P(n) \) is true \( \forall n \in \mathbb{N} \).

(b) The inequality \( P \geq R \) follows easily from \( S \geq P \) by replacing \( b_1 \leq b_2 \leq \ldots \leq b_n \)

by \( -b_n \geq -b_{n-1} \geq \ldots \geq -b_1 \).

Note:

(a) If \( a_i \)'s are strictly increasing, then equality holds \( S = P = R \) if and only if the \( b_i \)'s are all equal.

(b) Unlike most inequalities, we do not require the numbers involved to be positive.

Corollary 1

Let \( a_1, a_2, \ldots, a_n \) be real numbers and \( c_1, c_2, \ldots, c_n \) be its permutation. Then

\[ a_1^2 + a_2^2 + \ldots + a_n^2 \geq a_1 c_1 + a_2 c_2 + \ldots + a_n c_n \]

Corollary 2

Let \( a_1, a_2, \ldots, a_n \) be positive real numbers and \( c_1, c_2, \ldots, c_n \) be its permutation. Then

\[ \frac{c_1}{a_1} + \frac{c_2}{a_2} + \ldots + \frac{c_n}{a_n} \geq n \]

The rearrangement inequality can be used to prove many famous inequalities. Here are some of the highlights.
Arithmetic Mean - Geometric Mean Inequality (A.M. ≥ G.M.)

Let \( x_1, x_2, \ldots, x_n \) be positive numbers. Then \[ \frac{x_1 + x_2 + \ldots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n}. \]

Equality holds if and only if \( x_1 = x_2 = \ldots = x_n \).

Proof Let \( G = \sqrt[n]{x_1 x_2 \ldots x_n} \), \( a_1 = \frac{x_1}{G} \), \( a_2 = \frac{x_2}{G^2} \), \ldots, \( a_n = \frac{x_n}{G^n} = 1 \).

By corollary 2, \( n \leq \frac{a_1}{a_2} + \frac{a_2}{a_3} + \ldots + \frac{a_n}{a_1} = \frac{x_1}{G} + \frac{x_2}{G} + \ldots + \frac{x_1}{G} \iff \frac{x_1 + x_2 + \ldots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n} \)

Equality holds \( \iff \ a_1 = a_2 = \ldots = a_n \iff x_1 = x_2 = \ldots = x_n \).

Geometric Mean –Harmonic Mean Inequality (G.M. ≥ H.M.)

Let \( x_1, x_2, \ldots, x_n \) be positive numbers. Then \[ \sqrt[n]{x_1 x_2 \ldots x_n} \geq \frac{1}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}}. \]

Proof Define \( G \) and \( a_1, a_2, \ldots, a_n \) similarly as in the proof of A.M. – G.M.

By Corollary 2, \( n \leq \frac{a_1}{a_2} + \frac{a_2}{a_3} + \ldots + \frac{a_n}{a_1} = \frac{G}{x_1} + \frac{G}{x_2} + \ldots + \frac{G}{x_n} \) which then gives the result.

Root Mean Square - Arithmetic Mean Inequality (R.M.S. ≥ A.M.)

Let \( x_1, x_2, \ldots, x_n \) be numbers. Then \[ \sqrt[n]{x_1^2 + x_2^2 + \ldots + x_n^2} \geq \frac{x_1 + x_2 + \ldots + x_n}{n}. \]

Proof By Corollary 1, we cyclically rotate \( x_i \),

\[ x_1^2 + x_2^2 + \ldots + x_n^2 = x_1 x_1 + x_2 x_2 + \ldots + x_n x_n \]
\[ x_1^2 + x_2^2 + \ldots + x_n^2 \geq x_1 x_2 + x_2 x_3 + \ldots + x_n x_1 \]
\[ x_1^2 + x_2^2 + \ldots + x_n^2 \geq x_1 x_3 + x_2 x_4 + \ldots + x_n x_2 \]
\[ \ldots \]
\[ x_1^2 + x_2^2 + \ldots + x_n^2 \geq x_1 x_n + x_2 x_{n-1} + \ldots + x_n x_1 \]

Adding all inequalities together, we have \( n(x_1^2 + x_2^2 + \ldots + x_n^2) \geq (x_1 + x_2 + \ldots + x_n)^2 \)

Result follows. Equality holds \( \iff x_1 = x_2 = \ldots = x_n \).

Cauchy –Bunyakovskii – Schwarz inequality (CBS inequality)

Let \( a_1, a_2, \ldots, a_n \); \( b_1, b_2, \ldots, b_n \) be real numbers.

Then \( (a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)^2 \leq (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) \)

Proof The result is trivial if \( a_1 = a_2 = \ldots = a_n = 0 \) or \( b_1 = b_2 = \ldots = b_n = 0 \). Otherwise, define \( A = \sqrt{a_1^2 + a_2^2 + \ldots + a_n^2} \), \( B = \sqrt{b_1^2 + b_2^2 + \ldots + b_n^2} \)

Since both \( A \) and \( B \) are non-zero, we may let \( x_i = \frac{a_i}{A} \), \( x_{ni} = \frac{b_i}{B} \) \( \forall 1 \leq i \leq n \).
By Corollary 1,
\[ 2 = \frac{a_1^2 + a_2^2 + \ldots + a_n^2}{A^2} + \frac{b_1^2 + b_2^2 + \ldots + b_n^2}{B^2} \]
\[ \geq x_1x_{n+1} + x_1x_{n+2} + \ldots + x_n + x_2x_n + x_{n+1}x_1 + x_{n+2}x_1 + x_{n+2}x_2 + \ldots + x_{2n}x_n \]
\[ = 2(a_1b_1 + a_2b_2 + \ldots + a_nb_n) \]
\[ \Longleftrightarrow (a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2 \leq (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) \]
Equality holds \[ \Longleftrightarrow x_i = x_{n+i} \Longleftrightarrow a_iB = b_iA \quad \forall 1 \leq i \leq n. \]

**Chebyshev's inequality**

Let \[ x_1 \leq x_2 \leq \ldots \leq x_n \] and \[ y_1 \leq y_2 \leq \ldots \leq y_n \] be any real numbers.

Then
\[ x_1y_1 + x_2y_2 + \ldots + x_ny_n \geq \frac{(x_1 + x_2 + \ldots + x_n)(y_1 + y_2 + \ldots + y_n)}{n} \]
\[ \geq x_1y_n + x_2y_{n-1} + \ldots + x_ny_1 \]

**Proof**  By Rearrangement inequality, we cyclically rotate \( x_i \) and \( y_i \),
\[ x_1y_1 + x_2y_2 + \ldots + x_ny_n = x_1y_1 + x_2y_2 + \ldots + x_ny_n \geq x_1y_n + x_2y_{n-1} + \ldots + x_ny_1 \]
\[ \ldots \geq \ldots \geq \ldots \]
\[ x_1y_1 + x_2y_2 + \ldots + x_ny_n \geq x_1y_n + x_2y_{n-1} + \ldots + x_ny_1 = x_1y_n + x_2y_{n-1} + \ldots + x_ny_1 \]

Adding up the inequalities and divide by \( n \), we get our result.

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<th>Exercise</th>
<th>Hint</th>
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<td>1. Find the minimum of ( \frac{\sin^3 x}{\cos x} + \frac{\cos^3 x}{\sin x}, 0 &lt; x &lt; \frac{\pi}{2} )</td>
<td>Consider ( (\sin^3 x, \cos^3 x), \left( \frac{1}{\sin x}, \frac{1}{\cos x} \right) )</td>
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<tr>
<td>2. Proof: (i) ( a^2 + b^2 + c^2 \geq ab + bc + ca )</td>
<td>For (i) and questions below, Without lost of generality, let ( a \leq b \leq c ) Consider ( (a, b, c), (a^{n-1}, b^{n-1}, c^{n-1}) )</td>
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<td>3. Proof: ( \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq \frac{a + b + c}{abc} )</td>
<td>Consider ( \left( \frac{1}{a}, \frac{1}{b}, \frac{1}{c} \right), \left( \frac{1}{a}, \frac{1}{b}, \frac{1}{c} \right) )</td>
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<td>4. Proof: ( \frac{a}{b^2} + \frac{b}{c^2} + \frac{c}{a^2} \geq \frac{a + b + c}{abc} )</td>
<td>Consider ( \left( \frac{a}{b}, \frac{b}{c}, \frac{c}{a} \right), \left( \frac{a}{b}, \frac{b}{c}, \frac{c}{a} \right) )</td>
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<td>5. Proof: ( \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq a + b + c )</td>
<td>Consider ( \left( a^2, b^2, c^2 \right), \left( \frac{1}{a}, \frac{1}{b}, \frac{1}{c} \right) )</td>
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<td>6. Proof: If ( a, b, c &gt; 0 ) and ( n \in \mathbb{N} ) then ( \frac{a^n}{b+c} + \frac{b^n}{c+a} + \frac{c^n}{a+b} \geq \frac{a^{n-1} + b^{n-1} + c^{n-1}}{2} )</td>
<td>Consider ( \left( a^n, b^n, c^n \right), \left( \frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b} \right) )</td>
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<tr>
<td>7. Proof: If ( a, b, c &gt; 0 ), then ( a^ib^jc^k \geq (abc)^{\frac{a+b+c}{3}} )</td>
<td>Consider ( (a, b, c), (\log a, \log b, \log c) ) and use Chebyshev's inequality</td>
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