INSTRUCTIONS

1. This paper consists of Section A and Section B. Each section carries 50 marks.

2. Answer ALL questions in this paper. All working must be clearly shown.

3. All answers should be written in the answer book provided.

4. Unless otherwise specified, numerical answers must be either exact or correct to 4 decimal places.
Section A (50 marks)

1. When \((2x^5 + \frac{1}{x^3})^n\) is expanded in ascending powers of \(x\), the fourth term is a constant.
   (a) Find the value of \(n\) and the constant term.
   (b) Find the constant term of \((2x^5 - \frac{1}{x^3})^n\).

   (5 marks)

2. Let \(y = x^4 \sqrt{3x^2 + 2x + 3}\), where \(x > 0\).
   (a) Use logarithmic differentiation to express \(\frac{dy}{dx}\) in terms of \(x\) and \(y\).
   (b) Suppose \(w = 2^y\). Find the value of \(\frac{dw}{dx}\) when \(x = 1\).

   (5 marks)

3. In a shooting game, the probability that Jason hits the target is 75%.
   (a) If he shoots at the target 8 times, what is the probability that he hits the target at least 7 times?
   (b) He keeps shooting at the target until the target is hit 5 times. Find the probability that he shoots at
       the target exactly 10 times.

   (4 marks)

4. The number of mobile phones, \(X\), sold by a salesperson of a shop in a certain month has the following
   probability distribution.

   \[
   \begin{array}{c|c|c|c|c|c}
   x & 20 & 21 & 22 & 23 & 24 \\
   \hline
   P(X = x) & 0.11 & 0.28 & 0.31 & a & 0.3 - a
   \end{array}
   \]

   It is given that \(E(X) = 21.9\) and \(a\) is a constant.
   (a) Find the value of \(a\) and \(\text{Var}(X)\).
   (b) The basic monthly salary of the salesperson is $6000. For each mobile phone sold, the salesperson
       will get a commission of $200. Find the expected income of the salesperson in a month.

   (6 marks)
5. Let $X$ and $Y$ be two events with $P(X ∩ Y) = 0.51$ and $P(X ∩ Y') = 0.24$, where $Y'$ denotes the complement of the event $Y$.
   (a) Find $P(X)$.
   (b) It is given that $X$ and $Y$ are independent.
      (i) Find $P(Y)$.
      (ii) Find $P(X ∪ Y)$.

6. Suppose $\sum_{i=1}^{20} [x_i(x_i - 1)] = 112$ and $\sum_{i=1}^{20} x_i = 10.5$. Evaluate the following expressions:
   (a) $\sum_{i=1}^{20} x_i^2$
   (b) $\sum_{i=1}^{20} (x_i - 2)^2 + 1$

7. When a biased coin is tossed once, the probability of getting a head is $p$, where $0 < p < 1$. Now, the coin is tossed 20 times. It is given that the probability of getting at most 2 heads is $286p^2(1 - p)^{18}$.
   (a) Find the value of $p$.
   (b) If the coin is tossed $k$ times, find the least value of $k$ such that the probability of getting at least one head is greater than 0.95.

8. In Figure 1, $L$ is the tangent to the curve $C: y = 1 - 2x^5$ at the point $(k, 3)$.

   Figure 1

   (a) Find the value of $k$. Hence, find the equation of the tangent $L$.
   (b) Find the area of the shaded region bounded by $C$, $L$ and the y-axis in the figure.
9. In an experiment, the number \( p(t) \) (in thousands) of a certain kind of bacteria under controlled conditions can be modelled by

\[
p(t) = \frac{60}{0.5e^{10} - h} \quad (0 \leq t \leq 30),
\]

where \( t \) is the number of days elapsed since the beginning of the experiment, and \( h \) and \( k \) are constants.

(a) Express \( \ln \left( \frac{60}{p(t)} + h \right) \) as a linear function of \( t \).

(b) The graph of \( \ln \left( \frac{60}{p(t)} + h \right) \) against \( t \) has intercept 3.1 on the horizontal axis. The number of bacteria is 20 thousand at \( t = 0 \). Find the values of \( h \) and \( k \).

(c) It is given that the number of bacteria each day is more than that in the previous day. At least how many days after the beginning of the experiment will the number of bacteria exceed 90 thousand?

(7 marks)
Section B (50 marks)

10. The manager of a company wants to launch a promotion programme to increase the profit of product \( R \). The profit \( P \) (in ten thousand dollars) in the first month of sales can be modelled by

\[
P = 20e^{0.24} - 10A + k \quad (A \geq 0),
\]

where \( A \) (in ten thousand dollars) is the cost for the promotion programme and \( k \) is a constant. The manager finds that the profit of product \( R \) in the first month of sales will be 18 ten thousand dollars if there is no promotion programme.

(a) (i) Find the value of \( k \).

(ii) Find the least possible profit of product \( R \) in the first month of sales (correct your answer to the nearest ten thousand dollars).

(b) The manager claims that the greatest possible profit of product \( R \) in the first month of sales will be less than 20 ten thousand dollars if the cost for the programme is not more than 8 ten thousand dollars. Do you agree? Explain your answer.

(c) The manager eventually launches the promotion programme that offers free membership to customers who purchase product \( R \). There are no members before the start of the programme. The rate of change of the total number of members can be modelled by

\[
\frac{dN(t)}{dt} = 6e^{-0.2t} - e^{-0.1t} \quad (t \geq 0),
\]

where \( t \) is the number of weeks elapsed since the start of the programme and \( N(t) \) (in thousands) is the total number of members \( t \) weeks after the start of the programme.

It is given that the rate of change of the total number of members \( h \) weeks after the start of the programme is 1 thousand/week.

(i) Find the value of \( h \) (correct your answer to the nearest integer).

(ii) Using the approximate integral value of \( h \) obtained in (c)(i), find the total number of members \( h \) weeks after the start of the programme (correct your answer to the nearest thousand).
11. A scientist wants to investigate the number of fish in lakes $F$ and $G$. Let $P$ and $Q$ (in hundreds) be the numbers of fish in the lakes $F$ and $G$ respectively. It is given that the rates of change of $P$ and $Q$ can be modelled by

$$\frac{dP}{dt} = 27 - \frac{8t}{e^{3t}} \quad \text{and} \quad \frac{dQ}{dt} = \frac{8t + 16}{\sqrt{t + 1}}$$

respectively, where $t (0 \leq t \leq 9)$ is the number of years elapsed since the start of the research. It is given that the number of fish in each lake at $t = 0$ is 10 hundred.

(a) Consider the lake $F$. Figure 2 shows a sketch of the graph of $\frac{dP}{dt}$ against $t$ for $0 \leq t \leq 6$.

![Figure 2](image)

Figure 2

(i) Use the trapezoidal rule with six sub-intervals to estimate the number of fish (in hundreds) in the lake $F$ 6 years after the start of the research (correct your answer to 2 decimal places).

(ii) Is the estimate in (a)(i) an over-estimate or an under-estimate? Explain your answer.

(4 marks)

(b) Consider the lake $G$.

(i) Let $u = \sqrt{t + 1}$. Express $Q$ in terms of $t$.

(ii) Hence, find the number of fish (in hundreds) in the lake $G$ 6 years after the start of the research (correct your answer to 2 decimal places).

(6 marks)

(c) Using the results of (a) and (b), is it possible to determine which lake has a greater number of fish 6 years after the start of the research? Explain your answer.

(2 marks)
12. The monthly distances travelled by a car are normally distributed with mean $\mu$ km and standard deviation 40 km. A random sample of size 9 is drawn and the monthly distances (in thousand km) are shown below:

1.00 1.20 1.02 1.03 1.07 1.11 1.08 1.01 0.93

(a) (i) Find the sample mean.

(ii) Construct a 90% confidence interval for $\mu$.

(b) It is known that the cost of petroleum for driving a car for 1 km is $0.9. Assume the actual mean of monthly distance travelled is the same as the sample mean found in (a) (i).

(i) Find the mean and standard deviation of the monthly cost of petroleum.

(ii) Find the probability that the average monthly cost is greater than $975.

(iii) John claims that the average monthly cost of petroleum is $985. Do you agree? Explain your answer.
13. Assume that the number of customers that each counter handles in every 10 minutes in a supermarket check-out area is independent and follows a Poisson distribution with a mean of 3.5. A counter is classified as busy if it handles at least 4 customers in a period of 10 minutes.

(a) Find the probability that a counter is busy in a certain period of 10 minutes. (3 marks)

(b) A supervisor checks 4 counters in a certain period of 10 minutes. Find the probability that at least one busy counter is found. (2 marks)

(c) If 8 counters are open, find the probability that more than 5 of them are busy in a certain period of 10 minutes. (3 marks)

(d) Suppose 8 counters are open and one of them is randomly chosen. Find the probability that more than 5 of them are busy and the randomly chosen counter is not busy in a certain period of 10 minutes. (3 marks)

(e) The check-out area is said to be congested if more than 85% of the open counters are busy in a period of 10 minutes. Suppose 12 counters in the check-out area are open. A supervisor checks the counters in a certain period of 10 minutes. It is given that at more than 5 of the first 8 checked counters are busy. Find the probability that the check-out area is congested. (3 marks)

END OF PAPER
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Note: An entry in the table is the proportion of the area under the entire curve which is between $z = 0$ and a positive value of $z$. Areas for negative values of $z$ are obtained by symmetry.

![Diagram of the Standard Normal Curve](image)

$$A(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx$$